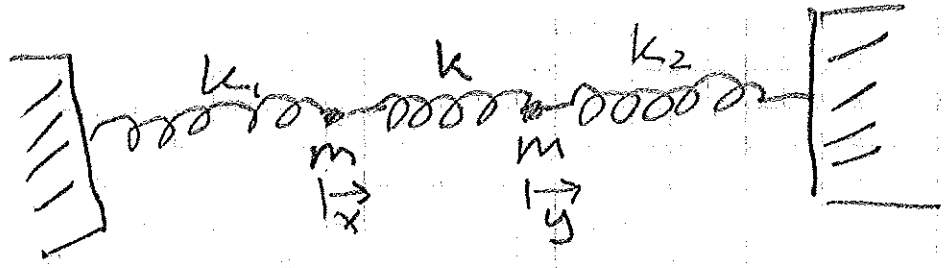
Coupled Oscillations

Coupled Harmonic Oscillators

Consider



$$m\ddot{x} + k_1 x + k(x-y) = 0$$

$$\rightarrow m\ddot{x} + (k_1+k)x - ky = 0$$

$$m\ddot{y} + (k_2+k)y - kx = 0$$

Define $\omega_1^2 \equiv \frac{(k_1+k)}{m}$, $\omega_2^2 \equiv \frac{(k_2+k)}{m}$, $q^2 \equiv \frac{k}{m}$

$$\rightarrow \quad \ddot{x} + \omega_1^2 x - q^2 y = 0$$

$$\ddot{y} + \omega_2^2 y - q^2 x = 0$$

Try solution of the form

$$x = ae^{i\omega t} \rightarrow \ddot{x} = -\omega^2 a e^{i\omega t}$$

$$y = be^{i\omega t} \rightarrow \ddot{y} = -\omega^2 b e^{i\omega t}$$

$$\rightarrow a(\omega_1^2 - \omega^2)e^{i\omega t} = bq^2 e^{i\omega t}$$

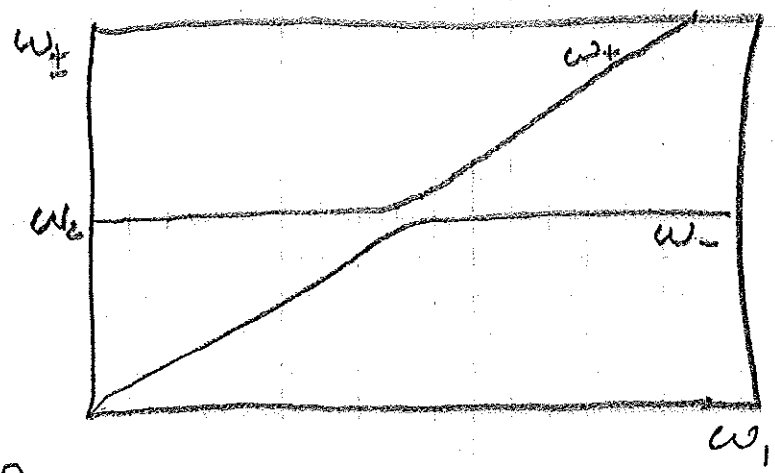
$$b(\omega_2^2 - \omega^2)e^{i\omega t} = aq^2 e^{i\omega t}$$

$$\rightarrow (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = q^4$$

$$\omega^4 - (\omega_1^2+\omega_2^2)\omega^2 + \omega_1^2\omega_2^2 - q^4 = 0$$

$$\rightarrow \omega^2 = \frac{(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\omega_1\omega_2 + q^4}}{2}$$

$$= \frac{(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + q^4}}{2}$$



If $\omega_1 = \omega_2 = \omega_0$, $(k_1 = k_2 \equiv k_0)$

$$\omega = \omega_0^2 \pm q^2 \quad : \frac{k_0}{m}, \frac{k_0+2k}{m}$$

More formally, we can write

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} + M\begin{pmatrix}x\\ y\end{pmatrix} = 0$$

If we assume $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix} e^{i\omega t}$

then $\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}\left(-\omega^2 I + M\right)\begin{pmatrix}a\\ b\end{pmatrix}$

In other words, ω^2 are the eigen values of $\underline{M}$ and the eigenvectors $\begin{pmatrix} a \\ b \end{pmatrix}$ are linear combinations of x and y which undergo simple harmonic motion

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Generalizing this to accelerators, if

$$\frac{\partial B_x}{\partial x} \neq \frac{\partial B_y}{\partial y} \neq 0$$

then

$$x' \propto -\frac{\partial B_y}{\partial x} x - \frac{\partial B_y}{\partial y} y$$

$$y' \propto \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial x} x$$

ie planes not independent

$\rightarrow$ motion will be coupled

If planes are coupled, then in general, we must write

$$\begin{pmatrix} x \\ x' \\ x \\ y' \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

In this representation, a normal quad with $\frac{1}{f} \equiv q = \frac{B'l}{(B\rho)}$ would be represented

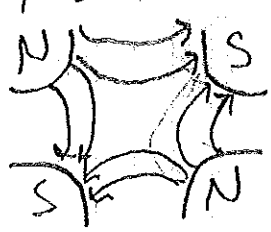
by
$$M_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q & 1 \end{pmatrix}$$

A pure skew quad is

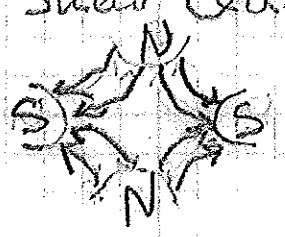
$$B_x = -\tilde{B}'x \rightarrow \Delta y' = \frac{+\tilde{B}'l}{(B\rho)} x \equiv +\tilde{q}\, x$$

$$B_y = -\tilde{B}y \rightarrow \Delta x' = \frac{+\tilde{B}l}{(B\rho)} y = +\tilde{q}\, y$$

Normal Quad



Skew Quad



So we write

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & +\tilde{q} & 0 \\ 0 & 0 & 1 & 0 \\ +\tilde{q} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix}$$

If we have a normal quad rotated by ϕ then (<u>home work</u>)

$$B_y = B'(x \cos 2\phi + y \sin 2\phi)$$

$$B_x = B'(-x \sin 2\phi + y \cos 2\phi)$$

For small angles $\cos\phi \sim 1$; $\sin\phi \sim \phi$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & -2q\phi & 0 \\ 0 & 0 & 1 & 0 \\ -2q\phi & 0 & q & 1 \end{pmatrix}$$

$$\Rightarrow M_{Q=q} M_{S=-2q\phi} \simeq M_{Q=q} + M_{S=2q\phi}$$

Coupling in Floquet Coords

we have

$$\Delta x'' = \tilde{q}\, y$$

$$\Delta x = 0 = \sqrt{\beta_x}\, \nu_x\, \Delta \xi_x$$

In Floquet coords (lecture 9)

$$X' = \frac{1}{\sqrt{\beta_x}\,\nu_x}\left(\dot{\xi}_x - \alpha\, \nu_x \xi_x\right)$$

$$\Delta x'' = \frac{\Delta \dot{\xi}_x}{\sqrt{\beta_x}\,\nu_x} = \tilde{q}\, y = \tilde{q}\sqrt{\beta_y}\, \xi_y$$

$$\rightarrow \Delta \dot{\xi}_x = \nu_x K\, \xi_y \; ; \; \Delta \dot{\xi}_y = \nu_y K \xi_x$$

where $K \equiv \tilde{q}\sqrt{\beta_x \beta_y}$

$\boxed{\text{Note: } \nu_x \neq \nu_y}$

Our motion is

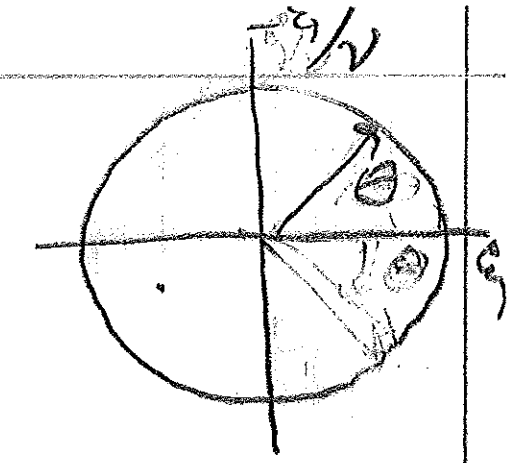
$$\xi_x = r_x \cos\Theta_x \; ; \; \xi_y = r_y \cos\Theta_y$$

$$\dot{\xi}_x = -\nu_x r_x \sin\Theta_x \; ; \; \dot{\xi}_y = -\nu_y r_y \sin\Theta_y$$

$$\rightarrow \Delta \dot{\xi}_x = \nu_x K r_y \cos\Theta_y$$

$$\Delta \dot{\xi}_y = \nu_y K r_x \cos\Theta_x$$

$$r^2 = \xi + \left(\frac{\dot{\xi}}{\nu}\right)^2$$



$$\rightarrow \Delta r^2 = \frac{2\dot{\xi}}{\nu^2}\Delta\dot{\xi}$$

$$= -\frac{2r\sin\Theta}{\nu}\Delta\dot{\xi}$$

$$\rightarrow \Delta r_x^2 = -2 r_x r_y K \sin\Theta_x \cos\Theta_y$$

$$\Delta r_y^2 = -2 r_x r_y K \sin\Theta_y \cos\Theta_x$$

<u>Phase changes</u> $\tan\Theta = -\frac{\dot{\xi}}{\nu\xi}$

$$\rightarrow \Delta\Theta = \frac{1}{1+\left(\frac{\dot{\xi}}{\nu\xi}\right)^2}\left(\frac{-\Delta\dot{\xi}}{\nu\xi}\right)$$

$$= -\frac{\xi}{r^2}\left(\frac{\Delta\dot{\xi}}{\nu}\right) = -\cos\Theta\,\frac{\Delta\dot{\xi}}{\nu r}$$

$$\rightarrow \Delta\Theta_x = -\frac{r_y}{r_x} K \cos\Theta_x \cos\Theta_y$$

$$\Delta\Theta_y = -\frac{r_x}{r_y} K \cos\Theta_x \cos\Theta_y$$

We assume one perturbation per turn, so

$$\frac{dr_x^2}{dn} = \Delta r_x^2 = -2 r_x r_y K \sin\Theta_x \cos\Theta_y$$

$$= 2 r_x \left(\frac{dr}{dn}\right)$$

$$\boxed{\begin{aligned}
\frac{dr_x}{dn} &= -r_y K \sin\Theta_x \cos\Theta_y \\
\frac{dr_y}{dn} &= -r_x K \sin\Theta_y \cos\Theta_x \\
\frac{d\Theta_x}{dn} &= 2\pi\nu_x - \frac{r_y}{2r_x} K \cos\Theta_x \cos\Theta_y \\
\frac{d\Theta_y}{dn} &= 2\pi\nu_y - \frac{r_x}{r_y} K \cos\Theta_x \cos\Theta_y
\end{aligned}}$$

Recall $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$\frac{dr_x}{dn} = -\frac{r_y}{2} K \left[\sin(\Theta_x + \Theta_y) + \sin(\Theta_x - \Theta_y)\right]$$

$$\frac{dr_y}{dn} = -\frac{r_x}{2} K \left[\sin(\Theta_x + \Theta_y) + \sin(\Theta_x - \Theta_y)\right]$$

$$\frac{d\Theta_x}{dn} = 2\pi\nu_x - \frac{r_y}{2r_x}\left[\cos(\Theta_x + \Theta_y) + \cos(\Theta_x - \Theta_y)\right]$$

$$\frac{d\Theta_y}{dn} = 2\pi\nu_y - \frac{r_x}{2r_y}\left[\quad '' \quad\right]$$

Difference Resonances

Focus on case where

$$\nu_x \approx \nu_y$$

The $\Theta_x + \Theta_y$ terms will oscillate quickly, so we focus on $\Theta_x - \Theta_y$

$$\rightarrow \quad \frac{dr_x}{dn} = -\frac{r_y}{2} K \sin(\Theta_x - \Theta_y)$$

$$\frac{dr_y}{dn} = +\frac{r_x}{2} K \sin(\Theta_x - \Theta_y)$$

$$\frac{d\Theta_x}{dn} = 2\pi\nu_x - \frac{r_y}{2r_x} K \cos(\Theta_x - \Theta_y)$$

$$\frac{d\Theta_y}{dn} = 2\pi\nu_y - \frac{r_x}{2r_y} K \cos(\Theta_x - \Theta_y)$$

We first note that

$$\frac{dr_x^2}{dn} = r_x \frac{dr_x}{dn} = -r_x r_y K \sin(\Theta_x - \Theta_y)$$

$$\frac{dr_y^2}{dn} = r_y \frac{dr_y}{dn} = +r_x r_y K \sin(\Theta_x - \Theta_y)$$

$$\rightarrow \quad \frac{dr_x^2}{dn} + \frac{dr_y^2}{dn} = \frac{d}{dn}(r_x^2 + r_y^2) = 0$$

so $\quad r_x^2 + r_y^2 = \epsilon_x + \epsilon_y = \text{constant}$

That is, emittance can exchange between two planes, but sum is constant

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Change variables to a rotating

$$\tilde{\Theta}_x \equiv \Theta_x - \pi(\nu_x + \nu_y)n \equiv \Theta_x - 2\pi\bar{\nu}n$$

$$\tilde{\Theta}_y \equiv \Theta_y - \pi(\nu_x + \nu_y)n \equiv \Theta_y - 2\pi\bar{\nu}n$$

In one rotation:

$$\Delta\tilde{\Theta}_x = 2\pi\nu_x - \pi(\nu_x + \nu_y)$$

$$= \pi(\nu_x - \nu_y) \equiv \pi\delta\nu$$

$$\Delta\tilde{\Theta}_y = 2\pi\nu_y - \pi(\nu_x - \nu_y)$$

$$= -\pi\delta\nu$$

equations become

$$\frac{dr_x}{dn} = -\frac{r_y}{2} K \sin\left(\tilde{\Theta}_x - \tilde{\Phi}_y\right)$$

$$\frac{dr_y}{dn} = \frac{r_x}{2} K \sin\left(\tilde{\Theta}_x - \tilde{\Theta}_y\right)$$

$$\frac{d\tilde{\Theta}_x}{dn} = \pi \delta\nu - \frac{r_y}{2r_x} K \cos\left(\tilde{\Theta}_x - \tilde{\Theta}_y\right)$$

$$\frac{d\tilde{\Theta}_y}{dn} = -\pi \delta\nu - \frac{r_y}{2r_x} K \cos\left(\tilde{\Theta}_x - \tilde{\Phi}_y\right)$$

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substitute

$$w_x \equiv r_x e^{i\tilde{\Theta}_x} \quad ; \quad w_y \equiv r_y e^{i\tilde{\Theta}_y}$$

$$\rightarrow \frac{dw_x}{dn} = \left(-\frac{r_y}{2} K \sin\left(\tilde{\Theta}_x - \tilde{\Theta}_y\right)\right) e^{i\tilde{\Theta}_x}$$

$$+ i r_x e^{i\tilde{\Theta}_x} \left(\pi \delta\nu - \frac{r_y}{2r_x} K \cos\left(\tilde{\Theta}_x - \tilde{\Theta}_y\right)\right)$$

$$= i w_x \pi \delta\nu$$

note flip!

$$- i \frac{r_y}{2} K \left(\cos\left(\tilde{\Theta}_y - \tilde{\Theta}_x\right) + i \sin\left(\tilde{\Theta}_y - \tilde{\Theta}_x\right)\right) e^{i\tilde{\Theta}_x}$$

$$= i\pi\delta\nu\, w_x - i\frac{K}{2} r_y e^{i\tilde{\theta}_y}$$

$$\frac{dw_x}{dn} = i\pi\delta\nu\, w_x - i\frac{K}{2} w_y$$

$$\frac{dw_y}{dn} = -i\pi\delta\nu\, w_y - i\frac{K}{2} w_x$$

Write as

$$\frac{dw_x}{dn} + i\pi\delta\nu\left(-w_x + \frac{K}{2\pi\delta\nu} w_y\right) = 0$$

$$\frac{dw}{dn} + i\pi\delta\nu\left(\frac{K}{2\pi\delta\nu} w_x + w_y\right) = 0$$

$$\rightarrow \frac{d}{dn}\begin{pmatrix} w_x \\ w_y \end{pmatrix} + i\pi\delta\nu\begin{pmatrix} -1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}\begin{pmatrix} w_x \\ w_y \end{pmatrix} = 0$$

$$\frac{d\vec{w}}{dn} + i\pi\delta\nu\, \mathbb{M}\, \vec{w} = 0$$

Define normal modes

$$\vec{u} = S\vec{w} \rightarrow \vec{w} = S^{-1}\vec{u}$$

$$\rightarrow \frac{d}{dn} S^{-1}\vec{u} + i\pi\delta\nu M S^{-1}\vec{u} = 0$$

multiply by S

$$\frac{d\vec{u}}{dn} + i\pi\delta\nu\left(S M S^{-1}\right)\vec{u} = 0$$

We see that if $\Lambda = S M S^{-1} = I$ is diagonal $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

then the solutions are

$$\vec{u} = \begin{pmatrix} u_{10} e^{-i\pi\delta\nu\lambda_1 n} \\ u_{20} e^{-i\pi\delta\nu\lambda_2 n} \end{pmatrix}$$

Solve using

$$M\vec{v}_n = \lambda_n I \vec{v}_n \rightarrow (M - \lambda_n I)\vec{v}_n = 0$$

eigenvector

$$\rightarrow \det\left(\underline{M} - \lambda_n \underline{I}\right) = 0$$

$$\underline{M} - \lambda \underline{I} = \begin{pmatrix} -(1+\lambda) & \varepsilon \\ \varepsilon & (1-\lambda) \end{pmatrix}$$

$$\rightarrow \lambda^2 - 1 - \varepsilon^2 = 0 \rightarrow \lambda = \pm\sqrt{1+\varepsilon^2}$$

$$\rightarrow U_1(n) = e^{-i\pi\delta\nu\sqrt{1+\varepsilon^2}\,n}$$

$$U_2(n) = e^{i\pi\delta\nu\sqrt{1+\varepsilon^2}\,n}$$

But remember we're still in a rotating system

$$\tilde{\Theta} \equiv \Theta - 2\pi\bar{\nu}$$

$$\rightarrow u = e^{-i2\pi\left(\bar{\nu} \mp \frac{\delta\nu}{2}\sqrt{1+\varepsilon^2}\right)n}$$

so we identify the new tunes to be

$$\nu_{1,2} = \bar{\nu} \mp \frac{\delta\nu}{2}\sqrt{1+\varepsilon^2}$$

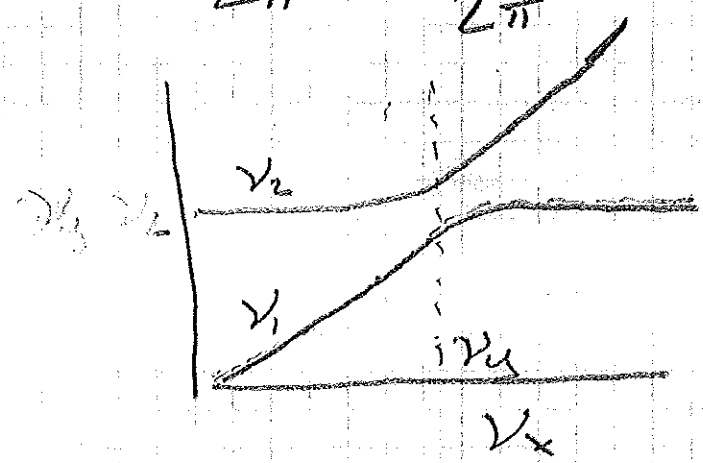
$$= \bar{\nu} \mp \frac{\delta\nu}{2}\sqrt{1+\frac{K^2}{4\pi^2\delta\nu^2}}$$

$$\boxed{= \bar{\nu} \mp \frac{1}{4\pi}\sqrt{K^2+4\pi^2\delta\nu^2}}$$

so even if $\nu_x = \nu_y \rightarrow \delta\nu^2 = 0$

we will have two tunes separated

by $\Delta\nu = \frac{K}{2\pi} = \frac{\sqrt{\beta_x\beta_y}\,\tilde{q}}{2\pi}$ ← skew term



The normal modes are given by (homework)

$$\vec{u} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \tan\alpha = \frac{1-\sqrt{1+\varepsilon^2}}{\varepsilon}$$

We can go back to our original coordinates with

$$\vec{w} = S^{-1}\vec{u}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

$$E_x = r_x^2 = |w_x|^2 = |\cos\alpha\, u_1 - \sin\alpha\, u_2|^2$$

$$E_y = r_y^2 = |w_y|^2 = |\sin\alpha\, u_1 + \cos\alpha\, u_2|^2$$

Remember, $u_i = u_0 e^{\mp i\pi\delta\nu\sqrt{1+\varepsilon^2}\,n}$

If we restrict ourselves to u_{10} or u_{20} real, then

$$r_x^2 = \varepsilon_x = u_{10}^2 \cos^2\alpha + u_{20}^2 \sin^2\alpha$$

$$- 2u_{10}u_{20}\cos\alpha\sin\alpha\left(e^{-i\pi\delta\nu\sqrt{1+\varepsilon^2}n} + e^{i\pi\delta\nu\sqrt{1+\varepsilon^2}n}\right)$$

$$= u_{10}^2\cos^2\alpha + u_{20}^2\sin^2\alpha$$

$$- 2u_{10}u_{20}\cos\alpha\sin\alpha\cos\left(2\pi\delta\nu\sqrt{1+\varepsilon^2}n\right)$$

$$r_y^2 = \varepsilon_y = u_{10}^2\sin^2\alpha + u_{20}^2\cos\alpha$$

$$+ 2u_{10}u_{20}\cos\alpha\sin\alpha\cos\left(2\pi\delta\nu\sqrt{1+\varepsilon^2}n\right)$$

Emittance will oscillate back and forth in $\dfrac{1}{\delta\nu\sqrt{1+\varepsilon^2}}$ turns!

Sum Resonances

Returning to our original equations on p. 9, but keeping the $\phi_x + \phi_y$ terms, we have

$$\frac{dr_x}{dn} = -\frac{r_y}{2} K \sin(\theta_x + \theta_y)$$ ← assume $\boxed{\nu_x + \nu_y \sim m}$

$$\frac{dr_y}{dn} = -\frac{r_x}{2} K \sin(\theta_x + \theta_y)$$

$$\frac{d\theta_x}{dn} = 2\pi\nu_x - \frac{r_y}{2r_x} K \cos(\theta_x + \theta_y)$$

$$\frac{d\theta_y}{dn} = 2\pi\nu_y - \frac{r_x}{2r_y} K \cos(\theta_x + \theta_y)$$

$$\frac{dr_x^2}{dn} = r_x \frac{dr_x}{dn} = -\frac{r_x r_y}{2} K \sin(\theta_x + \theta_y)$$

$$= \frac{dr_y^2}{dn}$$

$$\rightarrow \frac{dr_x^2}{dn} - \frac{dr_y^2}{dn} \rightarrow r_x^2 - r_2^2 = \text{constant}$$

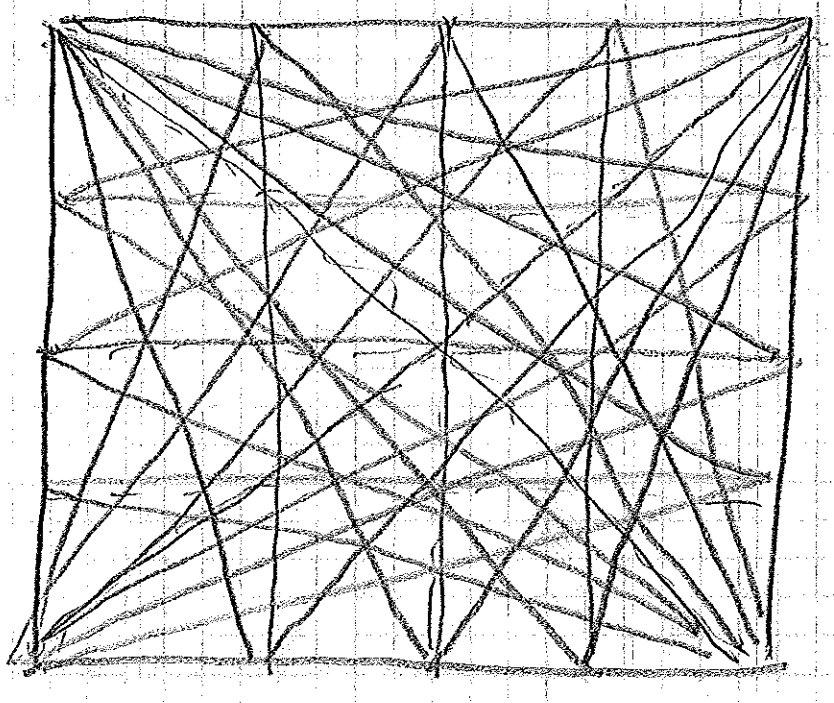
→ both can grow together!!

A complete analysis shows the eigenmodes will have the form

$$e^{\pm i\pi\sqrt{(\delta\nu)^2-\left(\frac{K}{2\pi}\right)^2}} \; ; \text{ where } \delta\nu \equiv m - (\nu_x + \nu_y).$$

$\rightarrow$ if $\delta\nu < \frac{K}{2\pi}$; there will be exponential growth!

Although we won't derive it in detail, we can see that if motion is <u>coupled</u> then we can analyze the system in terms of the normal coordinates

can do the same analyses we did in a single plane, so in general, a resonance can occur any time

$$\boxed{k_x \nu_x \pm k_y \nu_y = m}$$