Synchrotron Radiation

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For a relativistic particle, the total radiated power (S&E 8.1) is

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2 a^2}{c^3}\gamma^4 \quad . \quad a = \text{acceleration}$$

$$= \frac{v^2}{\rho} \sim \frac{c^2}{\rho}$$

$$= \frac{1}{6\pi\epsilon_0}\frac{e^2 c}{\rho^2}\gamma^4$$

In a magnetic field, $\rho = \frac{m\gamma}{eB}$

$$\rightarrow P = \frac{e^4}{6\pi\epsilon_0}\frac{B^2}{m_0 c^2}\gamma^2 = \frac{e^4}{6\pi\epsilon_0 (m_0 c^2)^3} B^2 E^2$$

Generally, a particle undergoing large β oscillations will radiate more power than one with small oscillations. Two effects compete

- Damping $\quad \tau_{\Delta E} \approx \underset{\text{period}}{\tau}\,\frac{E_s}{U_s}$ ← energy loss/t

- Quantum effects related to the statistics of the photons

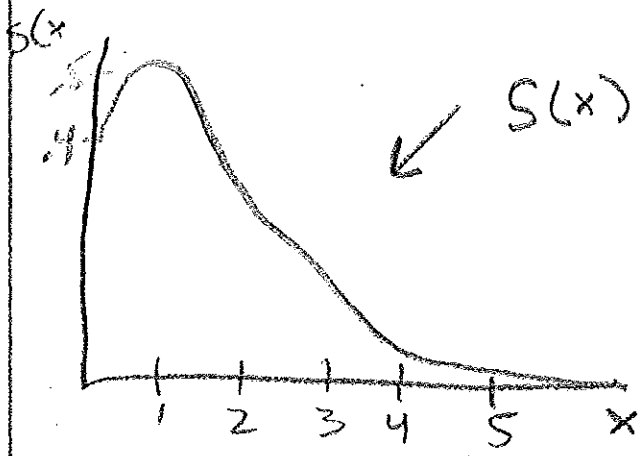
$$N_p = \dot{N}\tau \qquad \sigma_{\Delta E} = \sqrt{\dot{N}\tau_{\Delta E}\langle U^2\rangle}$$

↑ average γ energy

The power spectrum of radiation is

$$\frac{dP}{d\omega} = \frac{P}{\omega_c} S\left(\frac{\omega}{\omega_c}\right) \qquad \omega_c = \frac{3\gamma^3}{2}\frac{c}{\rho}$$

"critical frequency"



$$S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{5/3}(u)\,du$$

The power in an interval

$$\dot{n} = \frac{d\dot{W}}{du}$$

$$\frac{dP}{d\omega} d\omega = \dot{n}\hbar\omega\, du \,; \qquad d\omega = \frac{du}{\hbar}$$

$$\rightarrow \frac{dP}{d\omega}\frac{1}{\hbar} = \dot{n}\hbar\omega$$

$$\rightarrow \dot{n}(u) = \frac{1}{\hbar^2\omega}\frac{dP}{d\omega} = \frac{P}{(\hbar\omega)(\hbar\omega_c)} S\left(\frac{\omega}{\omega_c}\right)$$

$$= \frac{P}{u_c u} S\left(\frac{u}{u_c}\right)$$

$u_c \equiv \hbar\omega_c$

$$\dot{N} = \int_0^\infty \dot{n}(u)\,du = \frac{15\sqrt{3}}{8}\frac{P}{u_c}$$

The mean photon energy is

$$\langle u \rangle = \frac{P}{\dot{N}} = \boxed{\frac{8}{15\sqrt{3}}\,u_c}$$

The mean square of the energy is

$$\langle u^2 \rangle = \frac{1}{\dot{N}} \int_0^\infty u^2 \dot{n}(u)\,du$$

$$= \frac{P}{\dot{N}} \int_0^\infty \frac{u}{u_c}\, S\left(\frac{u}{u_c}\right) du$$

$$= \boxed{\frac{11}{27}\,u_c^2}$$

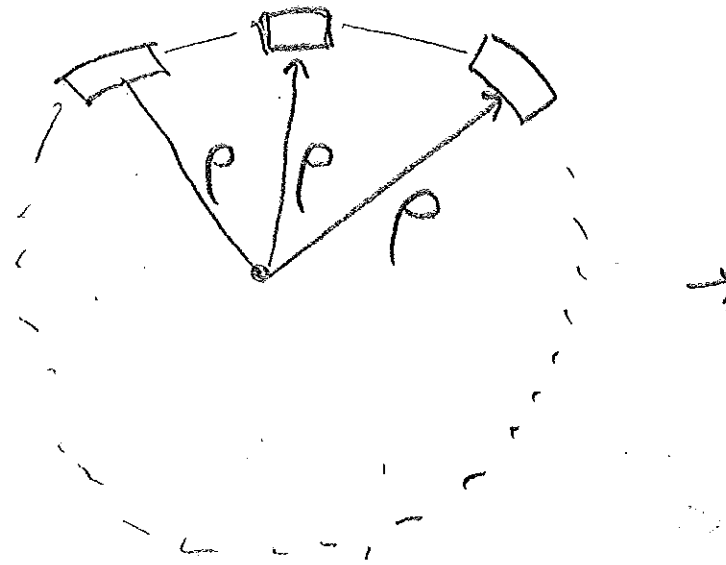
The energy lost per turn is

$$U_s = \oint P\,dt = \frac{e^2 c \gamma^4}{6\pi\epsilon_0} \int \frac{1}{\rho^2}\underbrace{\left(\frac{dt}{ds}\right)}_{=\frac{1}{c}} ds$$

$$= \frac{e^2 \gamma^4}{6\pi\epsilon_0} \int \frac{1}{\rho^2}\,ds$$

The RF system must recover this so $eV_0 \sin\phi_s = U_s$

For an iso magnetic ring



$$\Delta\theta = \frac{\Delta s}{\rho}$$

$$\rightarrow \oint \frac{ds}{\rho} = 2\pi$$

$$\rightarrow U_s = \frac{e^2 \gamma_s^4}{3\varepsilon_0 \rho}$$

= (for electrons)

$$U_s[\mathrm{MeV}] = .0885 \cdot \frac{E^4[\mathrm{GeV}]}{\rho[\mathrm{m}]}$$

$$U_c = \hbar\omega_c = \frac{3\gamma^3\hbar}{2}\frac{c}{\rho} =$$

$$U_c[\mathrm{keV}] = 2.218 \frac{E^3[\mathrm{GeV}]}{\rho[\mathrm{m}]}$$

For CESR $E = 5.29\ \mathrm{GeV}$

$\rho = 98\ \mathrm{m}$

$U_s = .71\ \mathrm{MeV}$

$u_c = 3.18\ \mathrm{keV}$

$\langle u \rangle = u_c \frac{8}{15\sqrt{3}} = .98\ \mathrm{keV}$

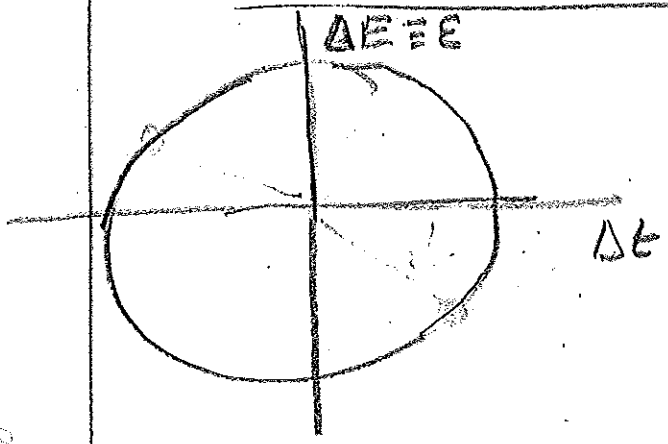
$\sqrt{\langle u^2 \rangle} = u_c \sqrt{\frac{11}{27}} = 2.0\ \mathrm{keV}$

The number of photons radiated in one turn is

$$\dot{N}_T = \frac{15\sqrt{3}}{8} \frac{P_\gamma}{u_c} \qquad P_\gamma = U_s$$

$$= \frac{15\sqrt{3}}{8} \frac{U_s}{u_c} = 721$$

Longitudinal Motion



$$\varepsilon = \varepsilon_0 \sin(2\pi \nu_s n + \delta)$$

$$\Delta t = \beta_L \varepsilon_0 (2\pi \nu_s n + \delta)$$

$$\frac{1}{\beta_L}(\Delta t)^2 + \beta_L \varepsilon^2 = \varepsilon_L = \text{invariant w/o radiation}$$

$$\frac{1}{\beta_L^2}(\Delta t)^2 + \varepsilon^2 = \varepsilon_0^2 \leftarrow \text{amplitude of energy oscillation}$$

(note confusing units)

If we radiate a photon of energy u

$$\rightarrow \varepsilon_{0,new}^2 = \frac{1}{\beta_L^2}(\Delta t)^2 + (\varepsilon - u)^2$$

$$= \frac{1}{\beta_L^2}(\Delta t)^2 + \varepsilon^2 - 2\varepsilon u + u^2$$

$$\rightarrow \Delta \varepsilon_0^2 = \varepsilon_{0,new}^2 - \varepsilon_0^2 = -2\varepsilon u + u^2$$

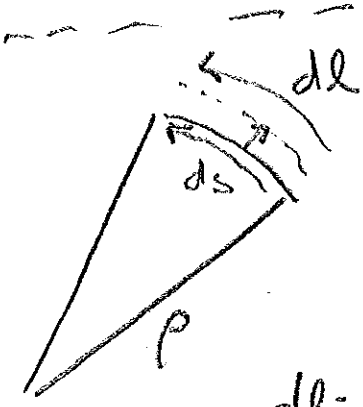
$$\frac{d\varepsilon_0^2}{dt} = -2\varepsilon \dot{N}\langle u \rangle + \dot{N}\langle u^2 \rangle$$

$$\left\langle \frac{d\varepsilon_0^2}{dt} \right\rangle = \frac{1}{\tau_s} \oint dt \, \frac{d\varepsilon_0^2}{dt}$$

$$= -\frac{2}{\tau_s} \underbrace{\oint \langle dt \, \varepsilon P \rangle}_{\text{where damping comes from}} + \frac{1}{\tau_s} \int dt \, \dot{N} \langle u \rangle^2$$

$$\oint \langle dt \, \varepsilon P \rangle$$

$$= \frac{1}{c} \oint \left\langle \left(1 + \frac{x}{\rho}\right) \varepsilon P \right\rangle ds$$



$$dl = ds\left(1 + \frac{x}{\rho}\right)$$

$$dt = \frac{dl}{c}$$

$$X = D \frac{\Delta p}{p} \approx D \frac{\varepsilon}{E_s}$$

$$\beta \sim 1$$

We have

$$P = \frac{e^4}{6\pi \varepsilon_0 (m_0 c^2)^3} B^2 E^2$$

$$= P(\varepsilon) = P_s \left(1 + 2 \frac{1}{B_0} \frac{dB}{dE} \varepsilon + 2 \frac{1}{E_s} \varepsilon\right)$$

Set dependence of field

$$\frac{dB}{dx} \quad B' \qquad \frac{dB}{dE} = \frac{dB}{dx}\frac{dx}{dE}$$

$$\equiv K(B\rho) \qquad = \frac{K(B\rho)D}{E_s}$$

$$\rightarrow P(\varepsilon) = P_s\left(1 + \frac{2\varepsilon}{E_s}\left(K\rho D + 1\right)\right)$$

Plugging into

$$\oint\langle dt\, \varepsilon P_s\rangle = \oint\langle ds\left(1+\frac{x}{\rho}\right)\varepsilon P_s\rangle$$

$$= \oint\left\langle ds\, P_s\left(1 + \frac{\varepsilon}{E_s}\frac{D}{\rho}\right)\varepsilon\left(1 + \frac{2\varepsilon}{E_s}\left(K\rho D+1\right)\right)\right\rangle$$

$$= \oint\left\langle ds\, P_s\left(\cancelto{0}{\varepsilon} + \frac{\varepsilon^2}{E_s}\left(2 + 2K\rho D + \frac{D}{\rho}\right) + \cancelto{0}{\varepsilon^3\frac{2D(K\rho D+1)}{E_s\rho}}\right)\right\rangle$$

$$\boxed{\begin{aligned} &\varepsilon = \varepsilon_0 \sin(2\pi\nu_s n + \delta) \\ &\rightarrow \langle\varepsilon\rangle = \langle\varepsilon^3\rangle = 0 \\ &\langle\varepsilon^2\rangle = \frac{\varepsilon_0^2}{2} \end{aligned}}$$

$$\rightarrow \oint \langle dt\, \varepsilon P \rangle = \frac{\varepsilon_0^2}{2E_s} \oint dt\, P_s \underbrace{\left[2 + 2K\rho D + \frac{D}{\rho} \right]}_{2U_s}$$

$$= \frac{\varepsilon_0^2 U_s}{E_s} + \frac{\varepsilon_0^2 U}{2E} \left(\frac{\beta^2 c \gamma^4}{6\pi \epsilon_0} \right) \underbrace{\oint dt \frac{1}{\rho^2} \left(2K\rho D + \frac{D}{\rho} \right)}$$

$$= \frac{1}{c} \oint ds \frac{1}{\rho^2} \left(2K\rho D + \frac{D}{\rho} \right)$$

$$= \frac{\mathcal{D}}{c} \oint ds \frac{1}{\rho^2}$$

where

$$\mathcal{D} \equiv \frac{\oint ds \frac{D}{\rho^2} \left(\frac{1}{\rho} + 2K\rho \right)}{\oint \frac{ds}{\rho^2}}$$

$$\rightarrow \boxed{\oint \langle dt\, \varepsilon P \rangle = \frac{\varepsilon_0^2 U_s}{2E_s} \left(2 + \mathcal{D} \right)} \; *$$

Going back to our original equation

$$\frac{d}{dt}(\mathcal{E}_0^2) = \underbrace{-\frac{\mathcal{E}_0^2 U_s}{\tau_s E_s}(2+\mathcal{D})}_{\text{damping}} + \underbrace{\frac{1}{\tau_s}\oint dt \langle \dot{N} u^2 \rangle}_{\text{growth from fluctuations}}$$

$$\rightarrow \mathcal{E}_0^2(t) = \mathcal{E}_0^2(0) e^{-t/\tau_{\mathcal{E}^2}} + \mathcal{E}_0^2(\infty)\left(1 - e^{t/\tau_{\mathcal{E}^2}}\right)$$

where $\frac{1}{\tau_{\mathcal{E}^2}} = \frac{U_s}{\tau_s E_s}(2+\mathcal{D})$

$$\mathcal{E}_0(\infty) = \frac{\tau_{\mathcal{E}^2}}{T_s}\oint dt \langle \dot{N} u^2 \rangle$$

so the energy decays in a time

$$\frac{1}{\tau_{\mathcal{E}}} = \frac{1}{2\tau_{\mathcal{E}^2}} = \frac{U_s}{2\tau_s E_s}(2+\mathcal{D})$$

In a separated function lattice, there is no bend in quads, so

$K \neq 0 \rightarrow \rho \rightarrow \infty$

$$\rightarrow D = \frac{\oint \frac{D}{\rho^2} ds}{\oint \frac{ds}{\rho^2}} = \frac{\frac{1}{\rho^2} \oint \frac{D}{\rho} ds}{\frac{1}{\rho} \oint \frac{ds}{\rho}}$$

$= C\alpha_c$

$\frac{1}{\rho} \oint \frac{ds}{\rho} = 2\pi$

$$= \frac{C\alpha_c}{2\pi\rho} \ll 1$$

$$\rightarrow \boxed{\frac{1}{\gamma_E} = \frac{U_s}{\gamma_s P_s}}$$

The equilibrium energy spread will be

$$\sigma_\varepsilon^2 = \langle \varepsilon_{,\infty} \rangle = \frac{1}{2} e_{0,\infty}^2$$

$$= \frac{\tau_\varepsilon}{4\tau_s} \oint dt \langle \dot{N} u^2 \rangle$$

Use $\dot{N} = \frac{15\sqrt{3}}{8} \frac{P}{u_c}$, $\langle u^2 \rangle = \frac{11}{27} u_c^2$

$$u_c = \frac{3}{2} \frac{\hbar \gamma^3}{\rho} c$$

$$\rightarrow \oint dt \langle \dot{N} u^2 \rangle = \frac{55}{16\sqrt{3}} \frac{e^2 \hbar c \gamma^7}{6\pi\varepsilon_0} \oint ds \frac{1}{\rho^3}$$

$$\rightarrow \sigma_{\varepsilon,\infty}^2 = \frac{E}{U_s (2+D)} \frac{55}{32\sqrt{3}} \left(\frac{e^2 \hbar c \gamma^7}{6\pi\varepsilon_0} \right) \oint \frac{ds}{\rho^2}$$

Transverse Effects of Synchrotron Radiation

1. Damping of synchrotron oscillation in both planes

2. Quantum excitation of horizontal β oscillations (through dispersion) leading to equilibrium emittance)

Two important results:

1. Robinson's Theorem:

$$\frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{2U_s}{E_s T_s}$$

energy loss/turn *

period

transverse damping times

For a separated function lattice

$$\tau_x = \tau_y$$

$$\tau_\varepsilon = \frac{E_s T_s}{U_s} \rightarrow \tau_x = \tau_y = 2\tau_\varepsilon = \frac{2E_s T_s}{U_s}$$

2. The equilibrium horizonal emittance is

$$\sigma_x = \sqrt{\underbrace{N}_{\text{photons emitted in damping time}}\tau_x}\,\frac{\sqrt{\langle U^2\rangle}}{E_s}\langle D\rangle$$

rms photon energy *

average dispersion

Here we go ...

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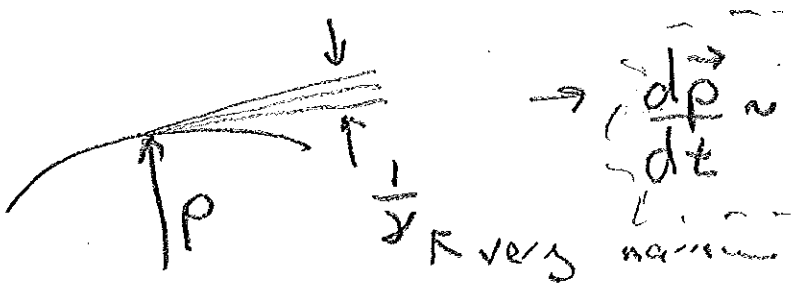
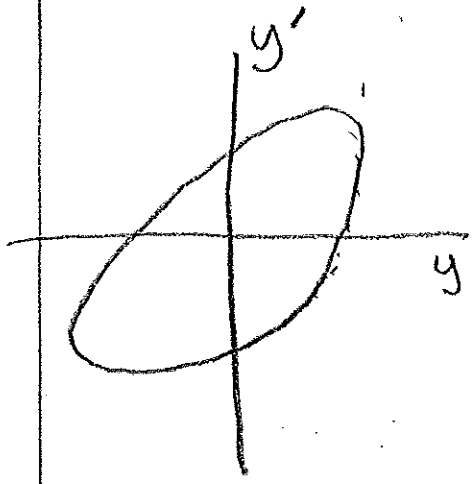


$$\rightarrow \frac{d\vec{p}}{dt} \sim -\frac{P}{c}\hat{\theta}$$

very narrow

radiated energy will reduce motion <u>along</u> the flight path!

For betatron motion in y plane



*

$$y = a\sqrt{\beta}\cos(\psi(s)+\delta)$$

$$y' = \frac{-a}{\sqrt{\beta}}\left(\alpha\cos(\psi(s)+\delta) + \sin(\psi(s)+\delta)\right)$$

Radiation does not affect y, but if we assume that the rf system <u>restores</u> <u>energy</u> <u>lost</u> <u>each</u> <u>turn</u> (U_s) then

Energy lost along path $\rightarrow \Delta y' = 0$

Energy added along $\hat{s} \rightarrow$ "adiabatic damping"

Recall

$$y' = \frac{P_y}{P} \qquad y' + \Delta y' = \frac{P_y}{P + \Delta_p} \approx y'\left(1 - \frac{\Delta p}{p}\right)$$

$$\rightarrow \frac{\Delta y'}{dn} = -y' \frac{\Delta p}{p} = -y' \frac{\Delta E}{E} = \boxed{-y' \frac{U_s}{E_s}} \quad *$$

turn

$$a^2 = \gamma y^2 + 2\alpha y y' + \beta y'^2$$

$$\frac{d a^2}{dy'} = 2\alpha y + 2\beta y'$$

$$\rightarrow \frac{d(a^2)}{dn} = -2(\alpha y + \beta y') y' \left(\frac{U_s}{E_s}\right)$$

$$= -\frac{2U_s}{E_s}\left(\alpha y y' + \beta y'^2\right)$$

$$= -\frac{2U_s}{E_s} a^2 \left[-(\alpha C^2 + S^2) + (\alpha^2 C^2 + 2\alpha SC + S^2) \right.$$

$$C \equiv \cos(\psi + \delta)$$
$$S \equiv \sin(\psi + \delta)$$

Every time the particle goes around, the phase will change, so we must average over random ψ

$$\langle SC \rangle = \left\langle \frac{1}{2} \sin(2(\psi+\delta)) \right\rangle = 0$$

$$\langle S^2 \rangle = \langle C^2 \rangle = \frac{1}{2}$$

$$\rightarrow \frac{d(a^2)}{dn} = -\frac{2U_s}{E_s} a^2 \left(\frac{1}{2}\right) = -a^2 \frac{U_s}{E_s} \; *$$

$$\frac{d(a^2)}{dt} = \frac{d(a^2)}{dn} \underbrace{\left(\frac{dt}{dn}\right)^{-1}}_{\gamma_s} = \boxed{-a^2 \frac{U_s}{\gamma_s E_s}} \quad (*)$$

$$y = a\sqrt{\beta} \cos(\psi + \delta) \rightarrow \langle y^2 \rangle = \beta \frac{a^2}{2}$$

$$\rightarrow \frac{d\sigma_y^2}{dt} = \frac{\beta}{2}\frac{da^2}{dt} = -\frac{\beta a^2}{2}\frac{U_s}{\tau_s E_s} = -\sigma_y^2 \frac{U_s}{\tau_s E_s}$$

$$\frac{d\sigma_y^2}{dt} = 2\sigma_y \frac{d\sigma}{dt} \rightarrow \frac{d\sigma}{dt} = -\sigma_y \frac{U_s}{2\tau_s E_s}$$

$$= -\frac{\sigma_y}{\tau_y}$$

$$\rightarrow \tau_y = 2\tau_s \frac{E_s}{U_s} \quad *$$

$$= 2\tau_\varepsilon$$

$\rightarrow$ damps to a very small value!

Horizontal Plane

Horizontal Motion is a bit more complicated because position depends on energy

$$X = X_\beta + D\frac{\mathcal{E}}{E_s}$$

ΔE

betatron motion

$$X' = X'_\beta + D'\frac{\mathcal{E}}{E_s}$$

where

$$X_\beta = a\sqrt{\beta}\cos(\psi+\delta) \equiv a\sqrt{\beta}\,C$$

$$X'_\beta = -\frac{a}{\sqrt{\beta}}\left(\alpha C + S\right)$$

$\sin(\psi+\delta)$

Now the radiated photon does not change X' or X', so

$$\Delta x = 0 = \Delta x_\beta - D\frac{u}{E_s}$$

energy of photon

$$\Delta x' = 0 = \Delta x'_\beta - D'\frac{u}{E_s}$$

We have

$$a^2 = \gamma x_\beta^2 + 2\alpha x_\beta x'_\beta + \beta x_\beta'^2$$

$$\Delta a^2 = \gamma\left(x_\beta + \frac{u}{E_s}D\right)^2 + 2\alpha\left(x_\beta + \frac{u}{E_s}D\right)\left(x'_\beta + \frac{u}{E_s}D'\right)$$

$$+ \beta\left(x'_\beta + \frac{u}{E_s}D'\right)^2$$

$$- \gamma x_\beta^2 - 2\alpha x_\beta x'_\beta - \beta x_\beta'^2$$

$$= 2\frac{u}{E_s}\Big[\alpha\left(D' x_\beta + D x'_\beta\right) + \gamma x_\beta D$$

$$+ \beta x'_\beta D'\Big]$$

$$+ \left(\frac{u}{E_s}\right)^2 \underbrace{\left[\gamma D^2 + 2\alpha D D' + \beta D'^2\right]}_{\equiv \mathcal{H}}$$

$$\frac{da^2}{dt} = \frac{2P}{E_s}\Big[\alpha\left(D' x_\beta + D x'_\beta\right) + \gamma x_\beta D + \beta D' x'\Big] \quad (*)$$

$$+ \dot{N}\left(\frac{u}{E_s}\right)^2 \mathcal{H}$$

$= \dot{N}u$

Averaging over one turn

$$\left\langle \frac{da^2}{dt} \right\rangle = \frac{1}{\tau_s} \oint dt \frac{da^2}{dt}$$

$$= \frac{2}{\tau_s E_s} \oint dt\, P\left(\alpha(D' x_\beta + D x') + \gamma D x_\beta + \beta D x'_\beta\right)$$

$$+ \frac{1}{\tau_s E_s^2} \oint dt\, \dot{N} u^2 \Big\}$$

Average over all all particles and phases

$$\frac{da^2}{dt} = \frac{2}{\tau_s E_s} \oint \Big\langle dt P\Big[\alpha(D' x_\beta + D x'_\beta) + \gamma D x_\beta + \beta D' x'_\beta\Big]\Big\rangle$$

$$+ \frac{1}{\tau_s E_s^2} \oint dt \langle \dot{N} u^2 \rangle \Big\}$$

$= A^*$

Use $dl = ds\left(1 + \frac{x}{\rho}\right)$; $c = \frac{dl}{dt}$

$$\rightarrow A = \frac{1}{c} \oint ds \Big\langle \left(1 + \frac{x_\beta}{\rho}\right) P\Big[\alpha(D' x_\beta + D x'_\beta) + \gamma D x_\beta + \beta D' x'_\beta\Big]$$

As before,

$$P = \frac{e^4}{6\pi\epsilon_0} \frac{B^2}{m_0 c^2} \gamma^2$$

$$\frac{dP}{dB} = \frac{2P}{B_0}$$

$$\rightarrow P(x_\beta) = P_s \left(1 + \frac{2}{B_0}\frac{dB}{dx} x_\beta\right)$$

$$= P_s\left(1 + 2K\rho x_\beta\right)$$

$$K \equiv \frac{B'}{(B\rho)}$$

$$A = \frac{1}{c}\oint ds \left\langle \left(1 + \frac{x_\beta}{\rho}\right) P_s \left(1 + 2K\rho x_\beta\right)\left[\alpha\left(D' x_\beta + D x_\beta'\right) + \gamma D x_\beta + \beta D' x_\beta'\right]\right\rangle$$

use $x = a\sqrt{\beta}\, C$

$y = -\frac{a}{\sqrt{\beta}}(\alpha C + S)$

as before $\langle C^2\rangle = \langle S^2\rangle = \frac{1}{2}$

$$\langle CS\rangle = 0$$

$$\rightarrow A = \frac{a^2}{2c} \oint dP_s \frac{D}{\rho}(1+2K\rho^2)$$

as before (p. 9)

$$= \frac{a^2 U_s}{2} \mathcal{D} \text{ ; where } \mathcal{D} \equiv \frac{\oint ds \frac{D}{\rho^3}(1+2K\rho^2)}{\oint \frac{ds}{\rho^2}}$$

Going back ...

$$\frac{da^2}{dt} = \frac{2}{\tau_s E_s} A + \frac{1}{\tau_s E_s^2} \oint dt \langle \dot{N} u \rangle \mathcal{H}$$

$$= \frac{a^2 U_s}{\tau_s E_s} \mathcal{D} + \frac{1}{\tau_s E_s^2} \oint dt \langle \dot{N} u^2 \rangle \mathcal{H}$$

But remember, we still have damping from $E_{RF} = U_s$ acceleration

$$\frac{da^2}{dt} = (\ldots) - a^2 \frac{U_s}{\tau_s E_s} \leftarrow \text{from p. 16}$$

$$\rightarrow \frac{da^2}{dt} = -\frac{a^2 U_s}{\tau_s E_0}(1-\mathcal{D}) + \frac{1}{\tau_s E_s^2} \oint dt \langle \dot{N} u \rangle \mathcal{H}$$

$$a^2(t) = a(0)^2 e^{-t/\tau} + a_\infty^2 \left(1 - e^{-t/\tau}\right)$$

$$\frac{1}{\tau} = \frac{U_s}{\tau_s E_s}(1 - D)$$

$$a_\infty^2 = \frac{\tau}{E_s^2 \tau_s} \oint dt \, \langle \dot{N} u^2 \rangle \mathcal{H}$$

as before $\sigma_x \propto \sqrt{a^2}$

$$\rightarrow \tau_x = 2\tau$$

$$\rightarrow \frac{1}{\tau_x} = \frac{U_s}{2\tau_s E_s}(1 - D)$$

$$\text{so } \frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{U_s}{2\tau_s E_s}(2 + D)$$

$$+ \frac{U_s}{2\tau_s E_s}(1 - D_s)$$

$$+ \frac{U_s}{2\tau_s E_s} = \frac{2U_s}{E_s \tau_s} \quad *$$

Robinson's Theorem

For small D

$$\tau_x = \tau_y = 2\tau_\varepsilon$$

$$\sigma_{x\infty}^2 = \langle x_\infty^2 \rangle = \frac{1}{2}\beta a^2$$

$$= \frac{\gamma_x \beta}{4 E_s^2 \tau_s} \oint dt \langle \dot{N} u^2 \rangle \mathcal{H}$$

$$= \frac{\beta}{2 U_s E_s (1-\mathcal{D})} \oint dt \langle \dot{N} u^2 \rangle \mathcal{H} \quad *$$

<u>Equilibrium emittance</u>

$$\varepsilon_x = \frac{\sigma_{x,\infty}^2}{\beta} = \frac{\oint dt \langle \dot{N} u^2 \rangle \mathcal{H}}{2 U_s E_s (1-\mathcal{D})}$$

Use $\dot{N} = \frac{15\sqrt{3}}{8} \frac{P}{u_c}$

$$\langle u^2 \rangle = \frac{11}{27} u_c^2$$

$$u_c = \frac{3}{2} \frac{\hbar \gamma^3 c}{\rho}$$

$$P = \frac{e^2 c \gamma^4}{6\pi\epsilon_0} \frac{1}{\rho^2}$$

$\rightarrow \langle \dot{N} u^2 \rangle$

$$= \left(\frac{15\sqrt{3}}{8}\right)\left(\frac{e^2 c \gamma^4}{6\pi\epsilon_0}\right)\left(\frac{1}{\rho^2}\right)\left(\frac{11}{27}\right)\left(\frac{3}{2}\hbar\gamma^3 c\right)\frac{1}{\rho}$$

$$= \frac{55}{16\sqrt{3}} \frac{e^2 \hbar c^2 \gamma^7}{6\pi\epsilon^0} \oint dt \, \mathcal{H}/\rho^3$$

$$= \frac{55\gamma^3 \hbar c}{16\sqrt{3}} \underbrace{\left(\frac{e^2 c \gamma^4}{6\pi\epsilon^0}\right) \oint dt \, \rho^2}_{U_s} \frac{\oint dt \frac{\mathcal{H}}{\rho^3}}{\oint dt \frac{1}{\rho^2}}$$

$dt = \frac{ds}{c}$

$$= \frac{55\gamma^3 \hbar c}{16\sqrt{3}} \frac{\oint ds \frac{\mathcal{H}}{\rho^3}}{\oint ds \frac{1}{\rho^2}}$$

$$\varepsilon_{x,\infty} = \frac{1}{2 U_s E_s (1-\mathcal{D})} \oint dt \langle \dot{N} \dot{u}^2 \rangle$$

$$= \frac{1}{2 U_s (mc^2\gamma)(1-\mathcal{D})} \frac{55\gamma^3 \hbar c U_s}{16\sqrt{3}} \frac{\oint ds \frac{\mathcal{H}}{\rho^3}}{\oint ds \frac{1}{\rho^2}}$$

$$\boxed{= C_q \frac{\gamma_s^2}{(1-\mathcal{D})} \frac{\oint ds \frac{\mathcal{H}}{\rho^3}}{\oint ds \frac{1}{\rho^2}} \qquad C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.8 \times 10^{-13} \text{ for } e}$$

For a separated function, isomagnetic machine

$$\rho = \rho_0 \quad \text{or} \quad \rho = \infty$$

$$\rightarrow \oint \frac{ds}{\rho^2} = \frac{1}{\rho_0} \int \frac{ds}{\rho} = \frac{2\pi}{\rho_0}$$

$$\rightarrow \boxed{\varepsilon_{x,\infty} = C_q \frac{\gamma^2}{2\pi\rho_0(1-D)} \oint ds \frac{\mathcal{H}}{\rho}} \quad (*)$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Approximate

$$\oint ds \frac{\mathcal{H}}{\rho} \sim \langle \mathcal{H} \rangle 2\pi$$

$$\sim \langle \gamma D^2 \rangle \sim \left\langle \frac{D^2}{\beta} \right\rangle$$

$$\langle D \rangle \approx \alpha_c R \qquad \langle \beta \rangle \sim \frac{R}{\nu_x}$$

$$\rightarrow \varepsilon_{0,\infty} \approx \gamma_s^2 \frac{C_q}{\rho} \left\langle \frac{D^2}{\beta} \right\rangle \approx C_q \gamma_s^2 \frac{R}{\rho} \alpha_c^2 \nu_x$$

$$\alpha_c = \frac{1}{\gamma_t} \sim \frac{1}{\nu_x} \qquad \approx \boxed{C_q \gamma_s^2 \frac{R}{\rho} \frac{1}{\nu_x^3}}$$