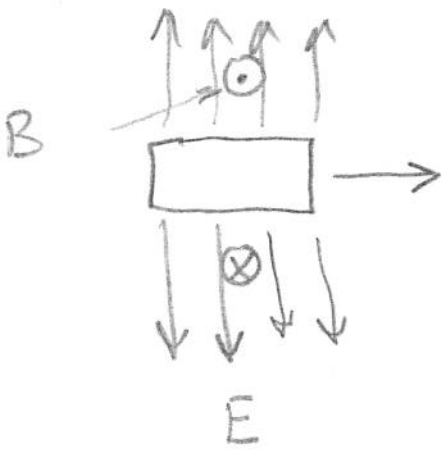
Collective Effects

Space Charge

Beam - Beam Interactions

So far, we have not considered the effect that particles in a bunch might have on each other, or on particles in another bunch.

Transverse space charge. Let's first consider transverse effect of space charge. Remember that relativistically, fields are "flattened"



$$\rho(r) = \frac{N_e}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

↑ radial charge density

If we look at the field at a radius r, we have

$$\oint \vec{E}\cdot d\vec{A} = E(2\pi r L) = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{N_e}{\sigma^2}\int_0^r r\,dr\, e^{-r^2/2\sigma^2}$$

$$\rightarrow E = \frac{N_e}{2\pi\varepsilon_0 r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$\rightarrow \vec{E} = \frac{Ne}{2\pi\varepsilon_0 r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{r}$$

Similarly, Ampere's Law gives

$$\oint \vec{B}\cdot d\vec{\ell} = 2\pi r B = \mu_0 I_{enc} \quad \swarrow = \frac{nev}{L}$$

$$= \mu_0 \frac{Nev}{2\pi\sigma^2 L}\int_0^r r\,dr\, e^{-r^2/2\sigma^2}$$

$$\rightarrow \vec{B} = \mu_0 \frac{Nev}{2\pi r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{\theta}$$

The force on a particle at radius r is

$$\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$= \frac{Ne^2}{2\pi r L}\left(1 - e^{-r^2/2\sigma^2}\right)\left(\frac{1}{\varepsilon_0}\hat{r} + \hat{s}\times\hat{\theta}\,\mu_0\right)$$

$\hat{s}\times\hat{\theta} = -\hat{r}$

$\mu_0 = \frac{1}{\varepsilon_0}(\varepsilon_0\mu_0) = \frac{1}{\varepsilon_0}\frac{1}{c^2}$

$$= \hat{r}\,\frac{Ne^2}{2\pi\varepsilon_0 r L}\left(1-e^{-r^2/2\sigma^2}\right)\left(1-\beta^2\right)$$

$$\frac{1}{\gamma^2} = \frac{1}{\gamma^2}$$

$$= \hat{r}\,\frac{Ne^2}{2\pi\varepsilon_0\gamma^2 r L}\left(1-e^{-r^2/2\sigma^2}\right)$$

$\vec{E}$ and $\vec{B}$ in same direction so in equation

$$|F_B| = q|B|v$$

$$|F_E| = qE \quad \rightarrow \text{In } x;\ B_y \rightarrow B_y - \frac{E_r}{v}$$

$$\text{In } y,\ B_x \rightarrow B_x + \frac{E_r}{v}$$

$$\rightarrow x'' = -\frac{1}{(B\rho)}\left(B_y - \frac{E_r}{v}\right)$$

$$= -\frac{e}{p}\left(B_y - \frac{E_r}{v}\right) = \frac{F}{vp}$$

$$y'' = \frac{e}{p}\left(B_x + \frac{E_y}{v}\right) = \frac{F_y}{vp}$$

$$\rightarrow F_x(x,y) = F_r \frac{x}{(x^2+y^2)^{1/2}}$$

$$= \frac{Ne^2}{2\pi\varepsilon_0 \gamma^2 L} \frac{x}{(x^2+y^2)}\left(1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$$

$$F_y(x,y) = \frac{Ne^2}{2\pi\varepsilon_0 \gamma^2 L} \frac{y}{(x^2+y^2)}\left(1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$$

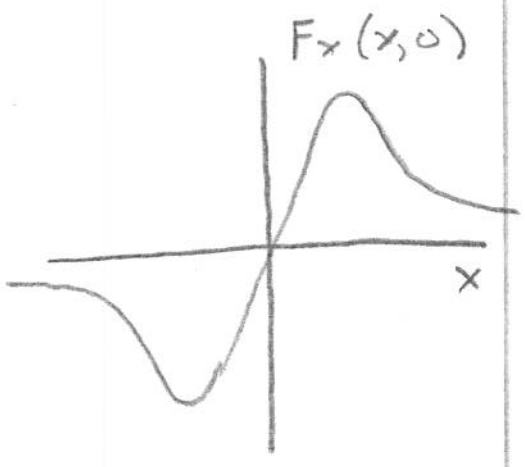
Non-linear and coupled → ouch!

but at small x and y

$$1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}} \sim \frac{(x^2+y^2)}{2\sigma^2}$$

$$\rightarrow F_x \approx x \frac{Ne^2}{4\pi\sigma^2\varepsilon_0\gamma^2 L}$$ ← ~ linear

$$F_y \approx y \frac{Ne^2}{4\pi\sigma^2\varepsilon_0\gamma^2 L}$$



$$x'' = \frac{F_x}{vp} = \frac{Ne^2}{4\pi\sigma^2\varepsilon_0 vp\gamma^2 L}\, x$$

$vp\gamma^2 = (\beta c)(\gamma\beta mc)\gamma^2$

$= \beta^2\gamma^3 mc^2$

$$= \frac{Ne^2}{4\pi\sigma^2\varepsilon_0\beta^2\gamma^3 c^2 L}\, x$$

$$= \frac{Nr_0}{\beta^2\gamma^2\sigma^2 L}$$ ← $r_0 \equiv \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$

$= 1.53\times10^{-18}$ m

This looks like a <u>defocussing</u> quad of strength

$$\frac{d\left(\frac{1}{f}\right)}{ds} \equiv -k = -\frac{Nr_0}{\beta^2\gamma^3\sigma^2 L}$$

So the tune shift is

$$\Delta \nu_x = \frac{1}{4\pi} \oint \beta_x(s) k \, ds$$

$$= -\frac{N r_0}{4\pi \beta^2 \gamma^3 L} \oint \frac{\beta_x}{\sigma^2} \, ds$$

$\sigma^2 = \varepsilon_x \beta_x$, so $\frac{\beta_x}{\sigma^2} = \frac{1}{\varepsilon_x}$

$\oint = 2\pi R / \varepsilon_x$

$$= -\frac{N r_0}{2\beta\gamma^2 L} \frac{R}{(\beta\gamma\varepsilon)}$$

$\varepsilon = \varepsilon_{x,N}$ (normalized emittance)

$$= -\frac{N r_0}{2\beta\gamma^2 L \varepsilon_{x,N}}$$

← maximum tune shift for particles near middle of beam

Example:

Fermilab Booster @ injection

$K = 400$ MeV

$N = 5 \times 10^{12}$

unbunched $\rightarrow$ $L = 2\pi R$

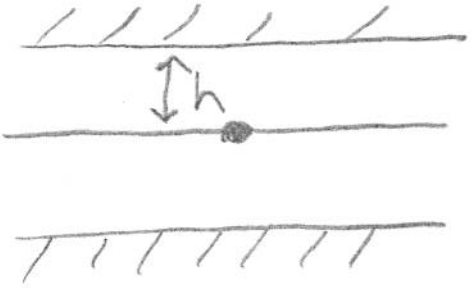
$\varepsilon_N = 2$ mm-mrad

$$\rightarrow \Delta\nu = -\frac{N r_0}{4\pi\beta\gamma^2\varepsilon_n} = -.247$$

↑ pretty large

Because this affects the tunes of individual particles, it's called an "incoherent" tune shift.

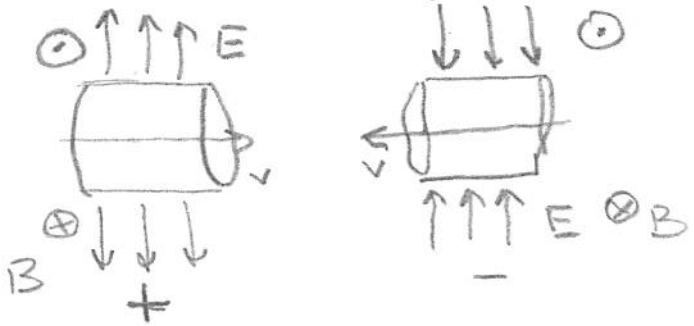
A coherent tune shift can be caused by image charges and currents in the walls



This is an important effect, but the resulting equations are quite complicated, and beyond the scope of this lecture

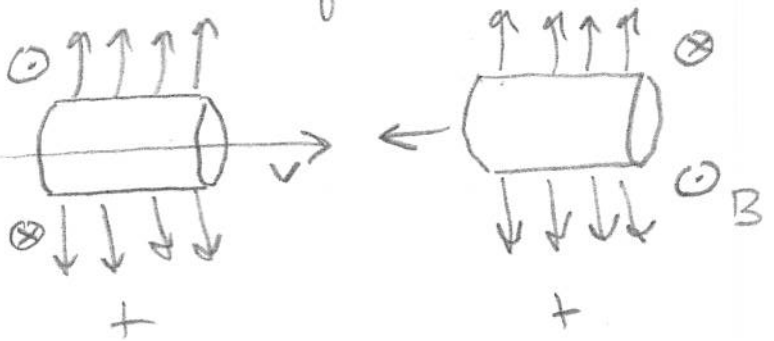
Beam-Beam Interaction

opposite sign particles



Both E and B fields attractive

same sign particles



Both E and B repulsive

In either case, the forces add so we use our derivation for incoherent tune shift (assume + and − particles)

$$\vec{F} = -\hat{r}\,\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1-e^{-r^2/2\sigma^2}\right)\underbrace{\left(1+\beta^2\right)}_{\approx 2}$$

$$= -\hat{r}\,\frac{Ne^2}{\pi\varepsilon_0 rL}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$x'' = \frac{F_x}{vp} \qquad y'' = \frac{F_y}{vp}$$

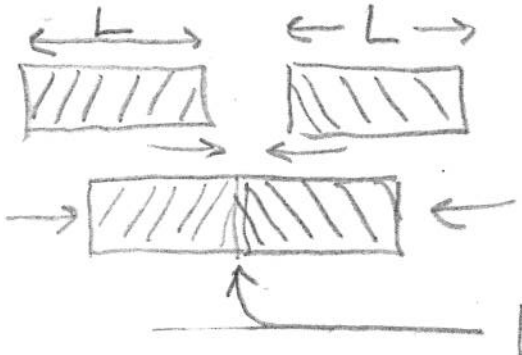
→ Integrate

$$\Delta x' = \frac{F_x}{vp}\Delta s \;;\; \Delta y' = \frac{F_y}{vp}\Delta s$$

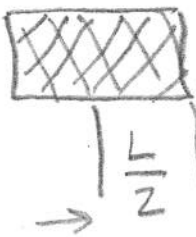
Tricky effect with colliders...

Effective Length

Two bunches



Front particles first encounter second bunch



first particles exit second bun

Effective Length = $\frac{L}{2}$

$$\rightarrow \Delta x' = \frac{F_x}{pv}\left(\frac{L}{2}\right)$$

$$= -\frac{Ne^2}{2\pi\varepsilon_0 r\,\gamma\beta^2 mc^2}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$\uparrow$ $\zeta \sim 1$

$$= -\frac{2r_0}{\gamma r}\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$\Delta' x = -\frac{2Nr_0}{\gamma} \frac{x}{(x^2+y^2)} \left(1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$$

$$\approx -\frac{Nr_0}{\gamma\sigma^2} x = -\frac{1}{f_{eff}} x$$

$\uparrow$ small x

$$\Delta y' \approx -\frac{Nr_0}{\gamma\sigma^2} y = -\frac{1}{f_{eff}} y$$

$$\rightarrow \Delta\nu = \frac{1}{4\pi} \frac{\beta^*}{f_{eff}} = \frac{Nr_0 \beta_x^*}{4\pi\sigma^2\gamma}$$

$$\equiv \xi_x \quad \text{"tune shift parameter"}$$

$$= \frac{Nr_0}{4\pi\varepsilon_n}$$

Luminosity and Tune Shift

There will generally be some tuneshift which will drive the beam onto a resonance and result in loss

The tune-shift parameter therefore becomes a limiting parameter.

We have the somewhat surprising result that

$$\xi = \frac{N r_0}{2\pi\varepsilon\gamma}$$

does <u>not</u> depend on β^*, but

only on $\frac{N}{\varepsilon} \equiv$ "brightness"

For a collider, we have the luminosity (for round beams)

revolution frequency → f

number of bunches → n_b

number in each bunch → N_1^2

$$\mathcal{L} = \frac{f\, n_b N_1^2}{4\pi\sigma^2}$$

σ^2 ← size

If we are "tuneshift limited", then we can express this as

$$\mathcal{L} = f \frac{n_b N \gamma}{r_0 \beta^*} \xi$$

We assume that β^* is already as small as possible, so we can win by

→ increasing ε and N proportionally keeping brightness constant.