Negative Mass Instability

Consider two particles in a bunch

Below transition

turn n

turn n+1

← farther apart

Above transition

turn n

turn n+1

← closer together

so the particles behave as if they had a "negative mass"

Let's examine this more closely

Consider a line density

$\lambda = \frac{q}{\ell}$

The fields outside the beam are given by

$$\int \vec{E}\cdot d\vec{A} = 2\pi r L E_r = \frac{1}{\epsilon_0} Q = \lambda L$$

$$\rightarrow \vec{E} = \frac{\lambda}{2\pi r \epsilon_0}\hat{r}$$

$$\int \vec{B}\cdot d\ell = 2\pi r B_\theta = \mu_0 I = \mu_0 \lambda v$$

$$\rightarrow \vec{B} = \frac{\mu_0 \lambda}{2\pi r}\hat{\theta}$$

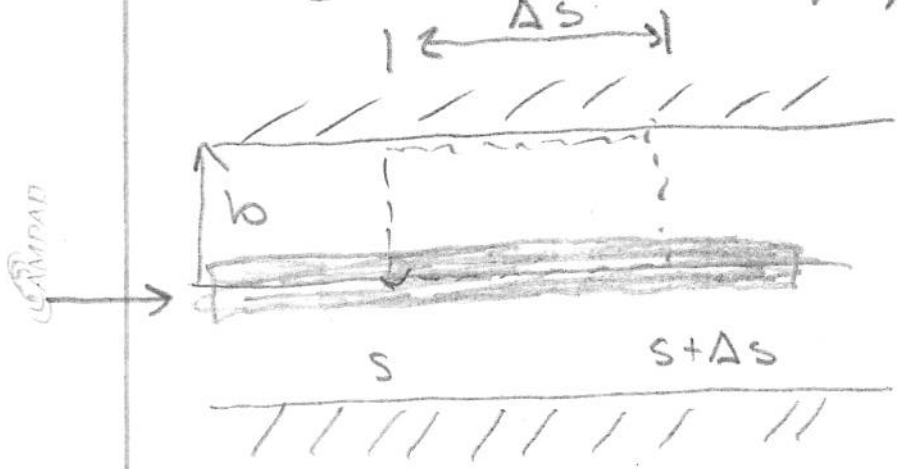
Inside the beam, the charge and current scale as $\frac{r^2}{a^2}$

$$\rightarrow \quad \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}\hat{r}\,; \quad \vec{B} = \frac{\lambda\mu_0}{2\pi}\frac{r}{a^2}\hat{\theta}$$

We now find the field along the beam axis, (E_s)

$$\text{Use} \quad \oint \vec{E}\cdot d\vec{\ell} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$$

Assume the beam is propagating through a beam pipe



$$\oint \vec{E}\cdot d\vec{l} = (E_s - E_w)\Delta s = \frac{d\lambda}{ds}\Delta s \equiv \lambda' \Delta s$$

$$+ \frac{\lambda(s+\Delta s) - \lambda(s)}{2\pi\epsilon_0}\left[\int_0^a \frac{r}{a^2} + \int_a^b \frac{1}{r}dr\right]$$

$$= \left[(E_s - E_w) + \frac{\lambda'}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\right]\Delta s$$

$$-\frac{\partial}{\partial t}\int B\cdot dA = -\frac{\partial}{\partial t}\frac{\mu_0 v \lambda}{2\pi}(\text{same integral})\Delta s$$

$$= -\Delta s\left[\frac{1}{2} + \ln\frac{b}{a}\right]\frac{\mu_0 \dot{\lambda} v}{2\pi} \qquad \dot{\lambda} = \frac{\partial\lambda}{\partial t}$$

Note that the shape of un-
perturbation is moving, so

$$\lambda(S) = \lambda(z - vt)$$

so $\frac{d}{dt}\lambda = \lambda'(-v)$

so $v\mu_0\dot{\lambda} = -v^2\mu\lambda'$

$$= -\left(\frac{v}{c}\right)^2 \frac{1}{\varepsilon^0}$$

so $-\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A} = \Delta s\left[\frac{\lambda'}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\beta^2\right]$

So combining the two sides, we

get $(E_s - E_w) = -\frac{\lambda'}{4\pi\epsilon_0}\underbrace{\left(1 + 2\ln\frac{b}{a}\right)}_{\equiv g}\underbrace{(1-\beta^2)}_{\frac{1}{\gamma^2}}$

$$= -\frac{\lambda' g_0}{4\epsilon_0\gamma^2}$$

If the wall is perfectly conducting, then $E_w = 0$, and we have

$$E_s = -\frac{g_0 \lambda'}{4\pi\epsilon_0 \gamma^2}$$

If the line density is perturbed then we can write

$$\lambda(\theta, t) = \lambda_0 + \lambda_1 e^{i(\Omega t - n\theta)}$$

↖ azimuthal location

↖ frequency of oscillation

n = mode number

General solution is a combination of these

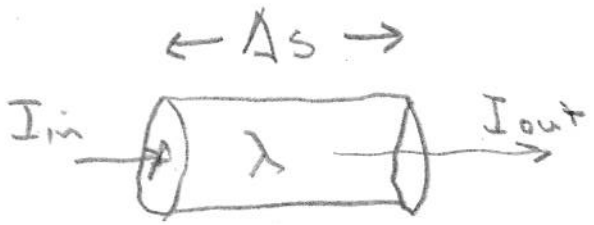
Mode will propagate with an angular frequency $\omega = \frac{\Omega}{n}$

↖ propagation around the ring

Remember, λ is a charge density, so it must satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\rightarrow \frac{d}{dt} Q_{enclosed} + - \int \vec{J} \cdot d\vec{A} = 0$$



$$Q = A \lambda \Delta s \qquad + I_{out} - I_{in}$$

$$\rightarrow \frac{dQ}{\partial t} = \dot{\lambda} \Delta s = \left(\lambda_{out} v_{out} - \lambda_{in} v_{in} \right)$$

$$= \left(\lambda_{out} \omega_{out} - \lambda_{in} \omega_{in} \right) R$$

$V = \omega R$ (R: ring radius)

$$= R \frac{\partial (\lambda \omega)}{\partial s} \Delta s$$

$$= RA \frac{\partial(\lambda\omega)}{\partial\theta} \frac{\partial\theta}{\partial s} \Delta s$$

$\frac{\partial\theta}{\partial s} \leftarrow \frac{1}{R}$

$$\rightarrow \frac{\partial\lambda}{\partial t} + \frac{\partial}{\partial\theta}(\lambda\omega) = 0$$

Look for a solution of the form

$$\omega(\theta,t) = \omega_0 + \omega_1 e^{i(\Omega t - n\theta)}$$

$$\rightarrow \lambda\omega = \lambda_0\omega_0 + (\lambda_0\omega_1 + \lambda_1\omega_0)e^{i(\Omega t - n\theta)} + \lambda_1\omega_1 e^{2i(\Omega t - n\theta)}$$

$\lambda_1\omega_1$ ← assume small

$$\rightarrow \frac{\partial}{\partial\theta}(\lambda\omega) \sim -in(\lambda_0\omega_1 + \lambda_1\omega_0)e^{i(\Omega t - n\theta}$$

$$\frac{\partial \lambda}{\partial t} = i\Omega_1 \lambda_1 e^{i(\Omega t - n\theta)}$$

$$\rightarrow i\Omega\lambda_1 - in(\omega_0\lambda_1 + \lambda_0\omega_1) = 0$$

$$\rightarrow \omega_1 = \frac{\Omega\lambda_1 - n\omega_0\lambda_1}{n\lambda_0}$$

$$= (\Omega - n\omega_0)\frac{\lambda_1}{n\lambda_0}$$

We can write the current as

$$I = R\omega\lambda = I_0 + I_1 e^{i(\Omega t - n\theta)}$$

$$= R\omega_0\lambda_0 + R(\omega_0\lambda_1 + \omega_1\lambda_0)e^{i(\Omega t - n\theta)}$$

$$\hookrightarrow = \frac{\Omega}{n}\lambda_1$$

$$\rightarrow I_0 = R\omega_0\lambda_0$$

$$I_1 = \frac{\Omega}{n} R\lambda_1$$

We now consider an individual particle in the distribution

$$\left(\frac{dw}{dt}\right)_p = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial \theta}\left(\frac{d\theta}{dt}\right)_p \quad (1)$$

$\left(\frac{dw}{dt}\right)_p$: particle; $\frac{\partial w}{\partial t}$: deformation; $\left(\frac{d\theta}{dt}\right)_p \approx \omega_0$

but the angular velocity of an individual particle is related to the time around the ring

$$\omega_p T = 2\pi$$

T: period

η: slip factor

$$\rightarrow \frac{\Delta \omega_p}{\omega_p} = -\frac{\Delta T}{T} = -\eta \frac{\Delta p}{p} = -\frac{\eta}{\beta^2}\frac{\Delta E}{E}$$

so we can write

$$\left(\frac{dw}{dt}\right)_p = -\frac{\eta \omega_0}{\beta^2 E}\frac{dE}{dt} \quad (2)$$

We're going to introduce the concept of a longitudinal impedance to characterize the energy lost per particle in terms of the total current, defined by

$$Z_{\parallel} = \frac{\text{energy lost/turn}}{(\text{unit charge})(\text{beam current})}$$

$$= \text{units of resistance}$$

Only interested in fluctuating part, so we write

$$\frac{dE}{dt} = \frac{dE}{dn}\left(\frac{\omega_0}{2\pi}\right) = -e\left(I_1 Z_{\parallel} e^{i(\Omega t - n\theta)}\right)\left(\frac{\omega_0}{2\pi}\right)$$

So equating ① and ②

$$\rightarrow \frac{\partial w}{\partial t} + \frac{\partial w}{\partial \theta}\omega_0 = \frac{e\eta I_1 Z_{\parallel}\omega_0^2}{2\pi\beta^2 E} e^{i(\Omega t - n\theta)}$$

Substitute

$$\omega = \omega_0 + \omega_1 e^{i(\Omega t - n\theta)}$$

and we get.

$$i\Omega\omega_1 - in\omega_1\omega_0 = \frac{e\eta I_1 Z_{\parallel}\omega_0^2}{2\pi\beta^2 E}$$

$$= i(\Omega - n\omega_0)\omega_1$$

but we showed $\omega_1 = (\Omega - n\omega_0)\frac{\lambda_1}{n\lambda_0}$

$$\rightarrow i(\Omega - n\omega_0)^2 \frac{\lambda_1}{n\lambda_0} \quad \swarrow \sim \frac{I_1}{I_0}$$

$$(\Omega - n\omega_0)^2 = -i\,\frac{e\eta I Z_{\parallel}\omega_0^2 n}{2\pi\beta^2 E}$$

motion is stable if RHS is

<u>real</u> and <u>positive</u>

$\rightarrow Z_{\parallel}\eta$ imaginary and > 0

local energy loss

$$= -E_s \Delta s$$

↑ field calculated earlier

$$= -\left(-\frac{g_0 \lambda'}{4\pi\epsilon_0 \gamma^2}\right)\Delta s$$

$$\underset{\text{set}}{=} I_1 Z_{11} e^{i(\Omega t - n\theta)} \frac{\Delta s}{2\pi R}$$ ← fraction of total

but $\lambda' = \frac{d\lambda}{d\theta}\frac{d\theta}{ds} = -\frac{in\lambda_1}{R} e^{i(\Omega t - n\theta)}$

$$-in\frac{\lambda_1 g_0}{4\pi\epsilon_0 \gamma^2 R} = \frac{I_1 Z_{11}}{2\pi R}$$ ← $I_1 = v\lambda_1$

$$\rightarrow \boxed{Z_{11} = -i\frac{n g_0}{2\epsilon_0 v \gamma^2}}$$ ← imaginary = good

want $\eta Z_{11} > 0$

so only stable below transition

→ unbunched beams bad above transition.