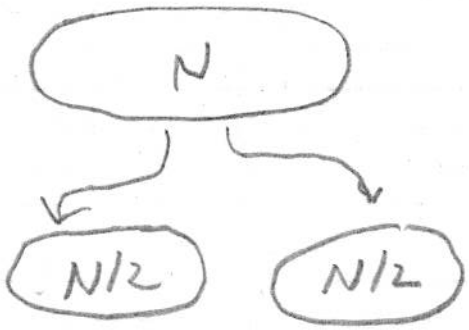
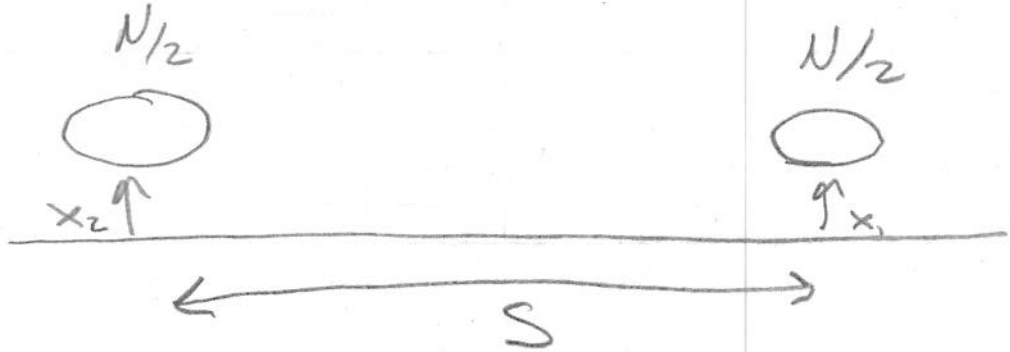
Macroparticle Models

A common approach to simple instabilities is to break a bunch into two "macrobunches"



The first problem we will apply this to is an electron linac.

At high γ, $\nu_s \to 0$ (ie no synchrotron motion), so the longitudinal relative positions of the particles remain fixed



we have

$$F_r = e\, Q_m\, m\, r^{m-1} \cos m\theta\, W_m(s)$$

Consider the lowest mode

$$Q_1 = \int \rho(r,\theta,z)\, r \cos\theta\, r\, dr\, d\theta\, dz$$

$$= \int \frac{N_e}{2}\, \delta(x-x')\,\delta(y)\,\delta(z-ct)\, x\, dx\, dy\, dz$$

$$= \frac{N_e}{2} x_1$$

$$\rightarrow F_x = F_r(\cos\theta = 1)$$

$$= e\, Q\, W_1(s)$$

$$= \frac{N_e^2}{2} W_1(s)\, x_1$$

If x_1 is executing β oscillations, then

$$x_1 = A_1 \cos \omega_\beta t$$

so the second particle sees

$$\ddot{x}_2 + \omega_\beta^2 x_2 = \frac{F_x}{m\gamma}$$ ← driving term

$$= \frac{Ne^2 \omega_1}{2m\gamma} x_1$$

$$= \frac{Ne^2}{2m\gamma} A_1 \cos \omega_\beta t$$

If the two have the same betatron frequency, then

the solution is

particular solution

$$x_2(t) = A_2 \cos \omega_2 t + p(t)$$

homogeneous solution

try $p(t) = K t \sin \omega_\beta t$

$$\dot{p}(t) = K \sin \omega_\beta t + K t \omega_\beta \cos \omega_\beta t$$

$$\ddot{p}(t) = K \omega_\beta \cos \omega_\beta t + K \omega_\beta \cos \omega_\beta t$$

$$- K t \omega_\beta^2 \sin \omega_\beta t$$

$$= 2 K \omega_\beta \cos \omega_\beta t$$

$$- K t \omega_\beta^2 \sin \omega_\beta t$$

Plug it in and we find

$$\overbrace{2K\omega_\beta \cos\omega_\beta t - K t\omega_\beta^2 \cancel{\sin\omega_\beta t} + \underbrace{\cancel{K\omega_\beta^2 t \sin\omega_\beta t}}_{\omega_\beta^2 x}}^{\ddot{x}} = A_1 \frac{Ne^2 W_1}{2m\gamma}\cos\omega_\beta t$$

$$\rightarrow K = A_1 \frac{Ne^2 W_1}{4\omega_\beta m\gamma}$$

$$\rightarrow x_2(t) = A_2 \cos\omega_\beta t + A_1 \frac{Ne^2 W_1}{4\omega_\beta m\gamma} t \sin\omega_\beta t$$

grows with time!

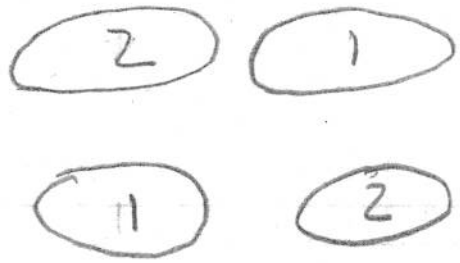
Turns out to be a problem in linacs. Amplitude grows as

$$\frac{Ne^2 W_1 L}{4\omega_\beta m\gamma L}$$ ← length of linac

This can lead to beam breakup in long linacs

Strong Head-Tail Intability

In a machine under-going synchrotron oscillations, we can imagine the two macroparticle changing places



$0 < t < T_s/2$

$\frac{T_s}{2} < t < T_s$

$\rightarrow$

$0 < t < \frac{T_s}{2}$: $\ddot{x}_1 + \omega_\beta^2 x_1 = 0$

$$\ddot{x}_2 + \omega_\beta^2 x_2 = \frac{N e^2 W_1}{2m\gamma} x_1$$

$\frac{T_s}{2} < t < T_s$: $\ddot{x}_1 + \omega_\beta^2 x_1 = \frac{N e^2 W_1}{2m\gamma} x_2$

$$\ddot{x}_2 + \omega_\beta^2 x_2 = 0$$

In unperturbed system

$$x = x_0 \cos \omega_\beta t + \frac{\dot{x}_0}{\omega_\beta} \sin \omega_\beta t$$

$$\dot{x} = \dot{x}_0 \cos \omega_\beta t - x_0 \omega_\beta \sin \omega_\beta t$$

Written in complex form

$$\tilde{x}(t) = x + \frac{i}{\omega_\beta} \dot{x} = \tilde{x} e^{-i\omega t}$$

We plug in the answer from last time

$$x_1 = A_1 \cos \omega_\beta t$$

$$x_2 = A_2 \cos \omega_\beta t + \frac{N e^2 W_1}{4 \omega_\beta m \gamma} t A_1 \sin \omega_\beta t$$

In complex form

$$\tilde{x}_1 = \tilde{x}_1(0) e^{-i\omega_\beta t}$$

$$\tilde{x}_2 = \tilde{x}_2(0) e^{-i\omega_\beta t} + i \frac{N e^2 W_1}{4 \omega_\beta m \gamma} t \tilde{x}_1(0) e^{-i\omega_\beta t}$$

get sin() term

We can express this in matrix form as

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{iNe^2W_1}{4\omega_\beta m\gamma}t & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} e^{-i\omega_\beta t}$$

From $0 < t < T_s/2$, we get

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i\eta_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta T_s/2}$$

$$\eta_1 = \frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma}$$

For the second half of the period, we get

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & i\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(T/2) \\ \tilde{x}_2(T/2) \end{pmatrix} e^{-i\omega_\beta T_s/2}$$

$$= \begin{pmatrix} 1 & i\eta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\eta & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta T_s}$$

$$= \begin{pmatrix} 1-\eta_1^2 & i\eta_1 \\ i\eta_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} e^{-i\omega_\beta T_s}$$

Assuming it undergoes this many times, we proved a long time ago that motion will only be stable if $|Tr(M)| \leq 2$

$$Tr(M) = 2 - \underbrace{\left(\frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma}\right)^2}_{\leq 4}$$

$$\rightarrow \boxed{\frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma} \leq 2} \leftarrow$$

"strong head tail instability"

We now consider the tune differences caused by chromaticity

$$\omega_\beta = 2\pi \nu f$$

revolution frequency

tune

If the momentum changes by $\delta \equiv \frac{\Delta p}{p}$, then

$$\omega_\beta(\delta) = 2\pi \nu(\delta) f(\delta)$$

$$= 2\pi (\nu_0 + \xi\delta) f_0 (1 - \eta\delta)$$

$$= 2\pi \nu_0 f_0 + 2\pi f_0 \xi \delta - 2\pi f_0 \nu_0 \eta \delta + O(\delta^2)$$

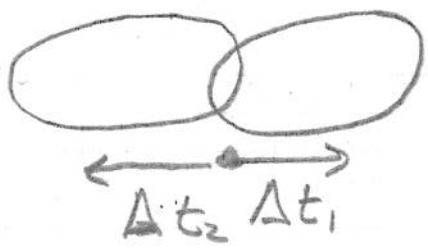
$$\approx \omega_\beta + \omega_0 \xi \delta$$

revolution angular frequency

We now write the positions of our macroparticles as

$$\Delta t_1 = \Delta t_0 \sin(\omega_s t)$$

$$\Delta t_2 = -\Delta t_0 \sin(\omega_s t)$$

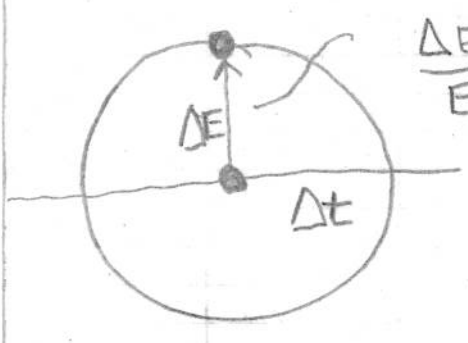


make s the independent variable

$$\rightarrow \Delta t_1 = \Delta t_0 \sin(\omega_s s/c)$$

$$\Delta t_2 = -\Delta t_0 \sin(\omega_s s/c)$$

We know that



$$\frac{\Delta E}{E} = \frac{\Delta p}{p} = \delta$$

$$\delta_{max} = \frac{\Delta t_{max}}{\tau_s}$$

We calculate the accumulated phase angle as

$$\phi = \int \omega_B(\delta)\,dt$$

$$= \frac{1}{c}\int \omega_B(\delta)\,ds$$

$$= \frac{1}{c}\int \frac{\omega_B\,ds}{c} + \omega_0\xi\int \frac{\delta\,ds}{c}$$

$$= \frac{\omega_B s}{c} + \omega_0\xi\int \delta\,dt$$

$$\uparrow \delta\eta\,dt$$

$$= \frac{\omega_B s}{c} - \frac{\omega_0\xi\Delta t_0}{\eta}\sin\frac{\omega_s s}{c}$$

So write

$$X_1(s) = \tilde{X}_1 e^{-i\left(\frac{\omega_B s}{c} - \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$X_2(s) = \tilde{X}_2 e^{-i\left(\frac{\omega_B s}{c} + \frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

↑ ie, slip is different

We identify the angular term as ω and write our equation as (for $0 < t < T_s/2$)

$$\ddot{x}_2 + \omega_2^2 x_x = \frac{F}{m\gamma}$$

$$\downarrow$$

$$c^2 \frac{d^2 x_2}{ds^2} + \left[\omega_\beta + \frac{\xi \omega_0 \Delta t_0 \omega_s}{\eta} \cos \frac{\omega_s s}{c}\right]^2 X_2$$

$$= \frac{N_e^2 W_1}{2m\gamma} X_1$$

Assume that the amplitude is changing slowly over time, we look at the first term

$$x_2 \simeq \tilde{x}_2 e^{-i\left[\frac{\omega_\beta}{c} s + \frac{\xi \omega_0}{\eta} \Delta t_0 \sin \frac{\omega_s s}{c}\right]}$$

$$c^2 \frac{d}{ds} x_2 = c^2 \left[\frac{d\tilde{x}_2}{ds} - i\left(\frac{\omega_\beta}{c} + \frac{\xi \omega_0}{\eta} \Delta t_0 \frac{\omega_s}{c} \cos \frac{\omega_s s}{c}\right) \tilde{x}_2\right]$$

$$\times e^{-i(\cdots)}$$

Assume $\frac{d^2}{ds^2}\tilde{x}_2 \approx 0$

$\rightarrow$

$$c^2 \frac{d^2}{ds} x_2 \approx c^2 \left[-2i \left[\frac{\omega_B}{c} + \frac{\xi \omega_0}{\eta} \Delta t_0 \frac{\omega_s}{c} \cos \frac{\omega_s s}{c} \right] \frac{d\tilde{x}_2}{ds} \right.$$

$$\left. - c^2 \left[\frac{\omega_B}{c} + \frac{\xi \omega_0}{\eta} \Delta t_0 \frac{\omega_s}{c} \cos \frac{\omega_s s}{c} \right]^2 \right]$$

$$= \frac{d\phi}{ds}$$

$$\times e^{-i\left[\frac{\omega_B}{c} s + \frac{\xi \omega_0}{\eta} \Delta t_0 \sin \frac{\omega_s s}{c}\right]}$$

$$\rightarrow c^2 \frac{d^2}{ds^2} x_2 + \left(\frac{d\phi}{dt}\right)^2 =$$

$$\tau = c^2 \left(\frac{d\phi}{ds}\right)^2$$

$$= c^2 \frac{d\tilde{x}_2}{ds} \left[-2i \left[\frac{\omega_B}{c} + \frac{\xi \omega_0}{\eta} \Delta t_0 \frac{\omega_s}{c} \cos \frac{\omega_s s}{c} \right] \right.$$

$$\times e^{-i\left[\frac{\omega_B}{c} s + \frac{\xi \omega_0}{\eta} \Delta t_0 \sin \frac{\omega_s s}{c}\right]} = \frac{N e^2 W_1}{2 m \gamma} x_1$$

If we assume $\omega_s \ll \omega_B$, this is approximately

$$c^2 \frac{d\tilde{x}_2}{ds}\left(-2i\frac{\omega_B}{c}\right)\cancel{e^{-i\left[\frac{\omega_B s}{c}+\frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right]}}$$

$$= \frac{N_e^2 W_1}{2m\gamma}\tilde{x}_1 \cancel{e^{-i\left[\frac{\omega_B s}{c}-\frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right]}}$$

$$\rightarrow \frac{d\tilde{x}_2}{ds} = \frac{iN_e^2 W_1}{4m\gamma\omega_B c}\tilde{x}_1 e^{i\left(2\frac{\xi\omega_0}{\eta}\Delta t_0 \sin\frac{\omega_s s}{c}\right)}$$

$$\approx \frac{iN_e^2 W_1}{4m\gamma\omega_B c}\tilde{x}_1\left(1+\frac{2i\xi\omega_0\Delta t_0}{\eta}\sin\frac{\omega_s s}{c}\right)$$

Integrate

$$\tilde{x}_2(s) = \tilde{x}_2(0) + \frac{iN_e^2 W_1}{4m\gamma\omega_B c}\tilde{x}_1\left[s+\frac{2i\xi\Delta t_0 c}{\eta\omega_s}\left(1-\cos\frac{\omega_s s}{c}\right)\right]$$

So now the to evolution over half a period is obtained by plugging in

$$S = \frac{T_s}{2} c = \frac{\pi c}{\omega_s}$$

$$\rightarrow \tilde{X}_2 = \tilde{X}_2(0) + \frac{i \pi N e^2 W_1}{4 m \gamma \omega_\beta c} \tilde{X}_1 \left(\frac{\pi c}{\omega_s} + i \frac{4 \xi \omega_0 \Delta t_0 c}{\eta \omega_s} \right)$$

$$\frac{\pi}{\omega_s} = \frac{T_s}{2}$$

$$= \tilde{X}_2(0) + \frac{i T_s N e^2 W_1}{8 m \gamma \omega_\beta} \left(1 + i \frac{4 \xi \omega_0 \Delta t_0}{\pi \eta} \right) \tilde{X}_1$$

$$\equiv \tilde{X}_2(0) + i \eta_1$$

Compare to our simple case, where

$$\eta_1 = \frac{N e^2 W_1 T_s}{8 \omega_\beta m \gamma}$$

and we see that we have added an imaginary part due to the

chromaticity, so we look at our previous matrix

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = e^{-i\omega_\beta T_s} \begin{pmatrix} 1-\eta_1^2 & i\eta_1 \\ i\eta_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ x_2(0) \end{pmatrix}$$

so, as before, stability requires

$$2 - \eta^2 = 0$$

We again write

$$\lambda_\pm = e^{\pm i\mu}$$

look at low intensid case

$$Tr(M) = 2\cos\mu = 2 - \eta_1^2$$

$$\rightarrow \mu = \pm \eta_1$$

so any imaginary part of η will lead to an exponential part of

$$e^{\pm i\mu} \rightarrow |\tilde{x}_2| = |x_2(0)| e^{\pm \frac{N e^2 W_1 \xi \omega_0 \Delta t_0 T_s t}{2 m \gamma \omega_\beta \pi \eta T_x}}$$

$$\rightarrow \frac{1}{\tau_{growth}} = \pm \frac{N e^2 W_1}{2\pi m \gamma \omega_B} \frac{\xi \omega_0 \Delta t_0}{\eta}$$

In fact, a more detailed analysis shows this is an overestimate, and chromaticity should be slightly positive after transition.