Landau Damping

For the case of the negative mass instability, we found that there was a threshold below which motion was stable. This is because there is a hidden damping mechanism in the dispersion relation

Consider a driven harmonic oscillator,

$$\ddot{x} + \omega_0^2 x = C \sin \omega t$$

$$\boxed{\begin{aligned} &\omega = \omega_0 \\ &\rightarrow x = \frac{C}{2\omega_0^2} \sin \omega_0 t \\ &\quad - \frac{C}{2\omega_0} t \cos \omega_0 t \\ &\quad \uparrow \text{grows} \end{aligned}}$$

$\omega \neq \omega_0$ General solution:

$$x = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega_0} \sin \omega t$$

$$+ p(t), \quad \text{where } \dot{p}(0) = p(0) = 0$$

Try $p(t) = A \sin \omega t + B \sin \omega_0 t$

$$\dot{p}(0) = A\omega \cos(0) + B\omega_0 \cos(0)$$

$$= 0 \rightarrow B = -\frac{\omega}{\omega_0} A$$

$$\ddot{x} = -A\left(\omega^2 \sin\omega t - \omega_0 \omega \sin\omega_0 t\right)$$

$$\rightarrow -A\left(\omega^2 \sin\omega t - \cancel{\omega_0 \omega \sin\omega_0 t}\right)$$

$$+ A\omega_0^2\left(\sin\omega t + \cancel{\frac{\omega}{\omega_0}\sin\omega_0 t}\right)$$

$$= A(\omega_0^2 - \omega^2)\sin\omega t \quad = C\sin\omega t$$

$$\rightarrow A = \frac{C}{(\omega_0^2 - \omega^2)}$$

$$\rightarrow x = \frac{C}{\omega_0^2 - \omega^2}\left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

$$= \frac{C}{(\omega_0 + \omega)\underbrace{(\omega_0 - \omega)}_{= -\delta\omega}}\left[\sin\omega t - \left(1 + \frac{\delta\omega}{\omega_0}\right)\sin\omega_0 t\right]$$

$$= \frac{C}{(\omega_0 + \omega)\omega_0}\sin\omega_0 t$$

$$- \frac{C}{(\omega_0 + \omega)\delta\omega}\left(\sin\omega t - \sin\omega_0 t\right)$$

$$= \frac{C}{(\omega_0+\omega)\omega_0} \sin\omega_0 t$$

$$\bar{\omega} = \frac{\omega+\omega_0}{2}$$

$$- \frac{C}{(\omega_0+\omega)\delta\omega}\left[\sin\left(\bar{\omega}+\frac{\delta\omega}{2}\right)t - \sin\left(\bar{\omega}-\frac{\delta\omega}{2}\right)t\right]$$

$$= \frac{C}{(\omega_0+\omega)\omega_0} \sin\omega_0 t$$

$\leftarrow$ small

$$- \frac{C}{(\omega_0+\omega)} t \, \frac{\sin\frac{1}{2}\delta\omega t}{\frac{1}{2}\delta\omega t} \cos\left(\frac{\omega+\omega_0}{2}t\right)$$

$\uparrow$ envelope $\qquad$ $\uparrow$ oscillation

$\rightarrow$ constant force does not produce an instability for $\omega \neq \omega_0$

Suppose we have

growing

$$\ddot{x} + \omega^2 x = C e^{i\omega_0 t} e^{t/\tau}$$

$$= C e^{(i\omega_0 + \frac{1}{\tau})t}$$

Try solution

$$x = A e^{(i\omega_0 + \frac{1}{\tau})t}$$

$$\rightarrow \ddot{x} = A\left(i\omega_0 + \frac{1}{\tau}\right)^2 e^{i\omega_0 t} e^{t/\tau}$$

$$\left(\cancel{\frac{1}{\tau^2}} + \frac{2\omega_0}{\tau} i - \omega_0^2\right) x$$

$\rightarrow 0$ assume

$$\frac{1}{\tau} \ll \omega_0$$

$$\rightarrow \left(\omega^2 - \omega_0^2 + \frac{2\omega_0}{\tau} i\right) A = C$$

$$\rightarrow \quad A = \frac{C}{(\omega^2 - \omega_0^2) + \frac{2i\omega_0}{\tau}}$$

for $\omega \approx \omega_0$, amplitude

$$\text{growth} \sim \frac{c}{2\omega_0 \delta\omega + 2i\omega_0/\tau}$$

phase shift

If there are an ensemble of particles, distributed in $\delta\omega$, then those in the range $\delta\omega = \pm \frac{1}{\tau}$ will participate in the growth.

Consider the case where the instability is driven by the particle displacement itself

$$\ddot{x} + \omega^2 x = k\langle x \rangle$$

If $\Delta\omega$ is the width of the distribution, then roughly $\frac{2}{\Delta\omega\tau}$

$$\rightarrow \ddot{x} + \omega_0^2 x = k\langle x \rangle \sim \frac{2k}{\Delta\omega\tau} x$$

$$\rightarrow \ddot{x} + \left(\omega_0^2 - \frac{2k}{\Delta\omega\tau}\right) x = 0$$

If the motion is to undergo growth, then the frequency $\omega^2 = \omega_0^2 - \frac{2}{\Delta\omega\tau} k$

must have a negative imaginary part; that is

$$\mathrm{Im}\{\omega\} = \mathrm{Im}\left\{\sqrt{\omega_0^2 - \frac{2}{\Delta\omega\tau}k}\right\} < 0$$

Assume $\frac{2k}{\Delta\omega\tau} \ll \omega_0^2$

so $\omega \sim \omega_0\left(1 - \frac{k}{\Delta\omega\omega_0^2\tau}\right)$

$$\rightarrow \mathrm{Im}\{\omega\} \approx -\frac{\mathrm{Im}(k)}{\Delta\omega\omega_0\tau} < 0$$

so for $\mathrm{Im}\{k\} > 0$, we have

$$e^{\frac{\mathrm{Im}(k)}{\Delta\omega\omega_0}\frac{t}{\tau}}$$

this is the argument of the exponential

so need $\mathrm{Im}\{k\} \approx \omega_0 \Delta\omega$

for the onset of instability

→ the larger the momentum spread, the larger the force coefficient needed to drive the instability

→ tune spread = stability

This is known as "Landau damping"