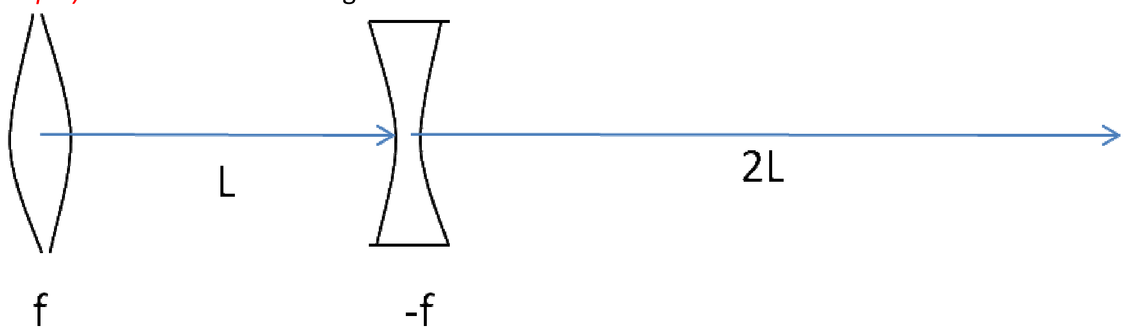
Accelerator Physics Exam
(with solutions-52 pts total)

1. *(10 pts)* Consider the following FODO cell



   a. Write the transfer matrix for this cell, assuming the particles are moving from left to right.

      (4 pts) The order of operation is from right to left, so the transfer matrix is the product of (drift 2L drift)(thin lens –f)(drift L)(thin lens f)

$$\boxed{\begin{aligned}
&\begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} \left(1-\frac{L}{f}\right) & L \\ -\frac{1}{f} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \left(1-\frac{L}{f}\right) & L \\ -\frac{L}{f^2} & \left(1+\frac{L}{f}\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(1-\frac{L}{f}-2\frac{L^2}{f^2}\right) & \left(3L+2\frac{L^2}{f}\right) \\ -\frac{L}{f^2} & \left(1+\frac{L}{f}\right) \end{pmatrix}
\end{aligned}}$$

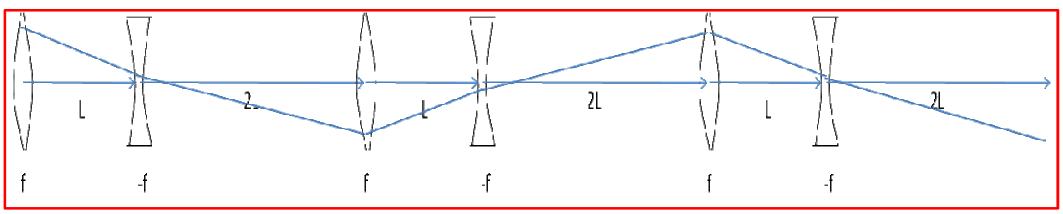
b. If a ring is made out of these FODO cells, express the condition for stable orbits *in this plane only* as an inequality between *L* and *f*; that is   or   , where *C* is a constant. Note, there might be more than one *mathematical* solution; be sure to specify a single inequality which is both physical and non-trivial.

(2 pts) The condition for stability is that the absolute value of the trace of the matrix be less than or equal to 2

$$
\begin{aligned}
&\left|\mathrm{Tr}(\mathbf{M})\right| = \left|2\cos\mu\right| \le 2 \\
&\Rightarrow \left|\left(1-\frac{L}{f}-2\frac{L^2}{f^2}\right)+\left(1+\frac{L}{f}\right)\right| = \\
&\Rightarrow \left|2-2\frac{L^2}{f^2}\right| \le 2 \\
&\Rightarrow -1 \le 1-\frac{L^2}{f^2} \le 1 \\
&\Rightarrow 0 \le \frac{L^2}{f^2} \le 2 \\
&\Rightarrow L \le \sqrt{2}f
\end{aligned}
$$

c. Assuming this condition is *just* met, sketch 3 cells and sketch approximately what a stable orbit would look like.

(2 pts) The maximum stable condition represents a phase advance of π, so the limiting orbit will move from a positive to a negative position at each cell. This is much like the limiting case of the simple FODO cell, except it will not pass through the center of the defocusing lens, so there will be a slight defocusing.



d. Now write the transfer matrix for the other plane (hint: this should be trivial at this point). Under the conditions described in (c), will motion be stable in this plane as well?

(2 pts) In the other plane, this cell has the focusing and defocusing lenses exchanged, which is the equivalent of simply changing the sign of f

$$\boxed{\begin{pmatrix} \left(1+\frac{L}{f}-2\frac{L^2}{f^2}\right) & \left(3L-2\frac{L^2}{f}\right) \\ -\frac{L}{f^2} & \left(1-\frac{L}{f}\right) \end{pmatrix}}$$

The trace of this matrix is the same as in the other plane, which means the stability condition will also be the same, so yes, motion will be stable in this plane as well.

2. *(10 pts)* A synchrotron consists of 28 identical cells. In the vertical plane, each cell has a phase advance of 62 degrees. The maximum beta function in each cell is 20 m, and the minimum is 10 m.

a. What is the tune of the machine?

(2 pts) If μ is the phase advance is in each cell, then

$$\nu = \frac{N\mu}{2\pi} = \frac{(28)(62)}{(360)} = 4.82$$

b. I wish to extract the beam by placing a kicker dipole in one cell and an extraction septum at the same location in the next. Would I choose the maximum or minimum beta points for this?

(2 pts) The equation relating the lateral deviation to an angular deflection is

$$\Delta y = \theta_0\sqrt{\beta_0\beta_1}\sin(\Delta\psi_{12})$$

If the kicker and septum are in the same places in their respective cells, then the two beta functions will be the same, and clearly I will get the maximum lateral deflection for a particular angular bend by choosing the maximum beta point in each cell.

c. Assuming I made the correct choice, what angular deflection would be required at the kicker to move the beam vertically by 4 cm at the location of the septum?

(2 pts) If both the bend dipole and the septum are at the maximum beta point in each cell, then

$$\Delta y = \theta_0\beta_{max}\sin(\mu)$$

$$\Rightarrow \theta_0 = \frac{(.04)}{(20)\sin(62)}$$

$$= .0023 = 2.3\text{ mrad} = .13°$$

d. If this is a 5 GeV electron beam, and the kicker is 1 m long, what magnetic field would be required to accomplish this?

(4 pts) Since this beam is very relativistic, $pc \approx E$, and the beam stiffness is

$$(B\rho) = \frac{5}{.3}$$

The bend angle of a dipole is approximately

$$\theta \approx \frac{Bl}{(B\rho)}$$

So we calculate the required field with

$$\begin{aligned} B &= \frac{\theta_0 (B\rho)}{l} \\ &= \frac{(.0023)(16.7)}{(1)} \\ &= .038 \text{ T} \end{aligned}$$

3. *(14 pts)* A proton beam is coasting (ie, not accelerating) in a synchrotron with the following parameters
   - Kinetic Energy: 100 GeV
   - Circumference: 5 km
   - Harmonic of the RF System: 500
   - Available RF voltage: 1 MV
   - Transition gamma (γ_T): 25

   Answer the following questions (assume ΔE, Δt, and ϵ_L are all defined in terms of the RMS values of the distributions):

   a. Calculate the period τ of the machine and the frequency (f_{RF}) of the RF system.

   (2 pts) We're going to need to know the energy and the relativistic γ and β for this beam.

$$E_s = K + mc^2 = 100.938 \text{ GeV}$$
$$\gamma = \frac{E_s}{mc^2} = \frac{100.938}{.938} = 107.6$$
$$\beta = \sqrt{1 - 1/\gamma^2} = .9999 \approx 1$$

The period is then

$$\tau = \frac{C}{\beta c} = \frac{(5000)}{(1)(3 \times 10^8)} = 16.7\ \mu\text{sec}$$

and the RF frequency is

$$f_{RF} = \frac{h}{\tau} = \frac{(500)}{16.7 \times 10^{-6}} = 30 \text{ MHz}$$

b. Calculate the slip factor (η) for this machine.

(2 pts) The slip factor is given by

$$\eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} = \frac{1}{25^2} - \frac{1}{107.6^2}$$
$$= .0015$$

c. What is the synchrotron tune (ν_s)

(2 pts) The equation for the synchrotron tune is

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{RF}\tau\eta}{E_s\beta^2}\cos\phi_s}$$

Since the machine is not accelerating, we know that

$$\sin\phi_s = 0$$

For motion to be harmonic, the product of the cosine and the slip factor must be negative. Since we're above transition, the slip factor is positive, so $\phi_s = \pi$, and the tune is (keeping eV_0 and E_s in units of GeV)

$$\nu_s = \frac{1}{2\pi}\sqrt{\frac{(.001)(2\pi \times 30 \times 10^6)(16.7 \times 10^{-6})(.0015)}{(100.938)}}$$
$$= .0011$$

d. What is the longitudinal beta (β_L) function (including units!)?

(2 pts) This is given by

$$\beta_L = \frac{\tau|\eta|}{2\pi E_s\beta^2\nu_s} = \frac{(16.7 \times 10^{-6})(.0015)}{2\pi(100.938 \times 10^9)(.0011)}$$
$$= 3.64 \times 10^{-17}\ \mathrm{s-eV^{-1}}$$
$$= 36.5\ \mathrm{ns-GeV^{-1}}$$

e. What is the maximum bucket height (ΔE_b) in eV which can be stored?

(2 pts) The equation for the maximum bucket height is

$$\Delta E_b = 2\frac{\sqrt{1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s}}{\omega_{RF}\beta_L}$$

for $\phi_s = \pi$, this becomes

$$\begin{aligned}\Delta E_b &= \frac{2}{\omega_{RF}\beta_L} \\ &= \frac{2}{(2\pi \times 30 \times 10^6)(3.64 \times 10^{-17})} \\ &= 2.91 \times 10^8 \text{ eV} \\ &= 291 \text{ MeV}\end{aligned}$$

f. Assuming that the bunch is matched, if the actual energy distribution (ΔE) is 1/5 of the maximum:

I. What is the longitudinal emittance (ϵ_L) in eV-s?

(2 pts) The emittance is related to the energy distribution by

$$\begin{aligned}\epsilon_L &= \beta_L(\Delta E)^2 \\ &= (3.64 \times 10^{-17})\left(\frac{2.91 \times 10^8}{5}\right)^2 \\ &= .124 \text{ eV} - \text{s}\end{aligned}$$

II. What is the time distribution (Δt) in s

(2 pts) The time distribution is related to the longitudinal emittance by

$$\begin{aligned}\Delta t &= \sqrt{\beta_L \epsilon_L} \\ &= \sqrt{(3.64 \times 10^{-17})(.124)} \\ &= 2.1 \times 10^{-9} \text{ s} \\ &= 2.1 \text{ ns}\end{aligned}$$

Alternatively, you could use

$$\begin{aligned}\Delta t &= \beta_L \Delta E \\ &= \beta_L \frac{\Delta E_b}{5} \\ &= (3.64 \times 10^{-17})\left(\frac{2.91 \times 10^8}{5}\right) \\ &= 2.1 \text{ ns}\end{aligned}$$

4. *(10 pts)* Recall from Homework 4

Machine	Particle type	Circumference	Bend radius of magnets	Beam Current	Energy
LEP	Electrons	27 km	3.5 km	5 mA	45 GeV
Tevatron	Protons	6.28 km	780 m	75 mA	1 TeV

When the Tevatron was shutting down, some people considered using the tunnel for a "giga-Z" factory: an electron-positron collider running on the Z_0 resonance (45 GeV/beam), but having 100 times the luminosity of LEP. We'll call it "Giga". Assume the following:

- Giga would be built in the Tevatron tunnel (ie, have the same circumference), and the new magnets would each have the same bend radius as the *present Tevatron*.
- The luminosity of LEP was limited by the beam-beam tune shift, and Giga would have the same maximum tune shift parameter and number of interaction points as LEP.
- All the lattice parameters at the collision point of Giga would be the same as LEP.
- You can assume that in each machine, the electron and positron beams are completely symmetric, in terms of beam and bunch size.
- Each machine is separated function and isomagnetic (all bend magnets have the same field).

Please answer the following:

a. What would be the required magnetic field for Giga, in Tesla?

(2 pts) The required magnetic field is just given by the beam stiffness over the desired bend radius

$$B_0 = \frac{(B\rho)}{\rho_0}$$
$$= \frac{\left(\frac{45}{.3}\right)}{780}$$
$$= .192 \text{ T}$$

b. What beam current would be required at Giga to achieve the 100 fold increase in luminosity?

(2 pts) The luminosity of a collider is

$$\mathcal{L} = \frac{f_{rev} n_b N_b^2 \gamma}{4\pi\beta^* \epsilon_N} R_{geom}$$

We know that the tune shift of the collider is proportional to the "brightness" (N_b/ϵ_N) (an explicit formula was given in lecture 14). We can therefore break this out (as we did when discussing the LHC luminosity), and we see.

$$\mathcal{L} = \frac{(f_{rev} n_b N_b)\gamma}{4\pi\beta^* \epsilon_N} \left(\frac{N_b}{\epsilon_N}\right) R_{geom}$$

So if the tune shift is the limit, and – as we specified - everything else about the machines is the same, then the luminosity will just depend on the product $(f_{rev} n_b N_b)$, which is simply the beam current over the charge. Therefore, to increase the luminosity by 100, we would need to increase the beam current by 100, to

$$500 \text{ mA}$$

c. What does this correspond to for the product of the number of bunches and bunch size $(n_b N_b)$ in Giga?

(2 pts) The current is given by

$$I = e f_{rev} n_b N_b$$

so

$$(n_b N_b) = \frac{I}{e f_{rev}} = \frac{I}{e}\frac{C}{c}$$

$$= \frac{(.5)(6280)}{(1.6 \times 10^{-19})(3 \times 10^8)}$$

$$= 6.5 \times 10^{13}$$

d. What would be the energy lost per particle per turn (in GeV) to synchrotron radiation in Giga[1]?

(2 pts) We derived the expression for energy loss/turn in the lecture on synchrotron radiation, but it's pretty easy to reproduce. For a particle being bent with a radius of curvature ρ, the radiated power is

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2 c\gamma^4}{\rho^2}$$
$$U_s = \int_0^\tau P dt = \frac{1}{c}\oint P ds$$
$$= \frac{e^2\gamma^4}{6\pi\epsilon_0}\oint \frac{1}{\rho^2} ds$$

But in a separated function, isomagnetic lattice, ρ is either ρ_0 or infinity, so we can write

$$U_s = \frac{e^2\gamma^4}{6\pi\epsilon_0}\frac{1}{\rho_0}\oint \frac{1}{\rho} ds$$
$$= \frac{e^2\gamma^4}{6\pi\epsilon_0}\frac{1}{\rho_0}(2\pi)$$
$$= \frac{e^2\gamma^4}{3\rho_0\epsilon_0}$$

(which we could have just copied from page 3 of the synchrotron radiation lecture), so

$$U_s = \frac{(1.6\times 10^{-19})^2\left(\frac{45000}{.511}\right)^4}{3(780)(8.85\times 10^{-12})}$$
$$= 7.43\times 10^{-11}\ \mathrm{J}$$
$$= .465\ \mathrm{GeV}$$

[1] If this turns out to be greater than the particle energy, it might still be OK, assuming we are continuously re-accelerating the particles between magnets.

e. What would be total power loss (in Watts) for *each* beam in Giga?

(2 pts) The power loss will be given by the energy loss per turn per particle multiplied by the total number of particles times the revolution frequency

$$
\begin{aligned}
P_{loss} &= (f_{rev} n_b N_b) U_s \\
&= \frac{c}{C}(n_b N_b) U_s \\
&= \frac{(3\times 10^8)}{6280}(6.5\times 10^{13})(7.43\times 10^{-11}) \\
&= .23\ \mathrm{GW}
\end{aligned}
$$

Of course we could save a step by realizing that the first equation is equivalent to

$$
\begin{aligned}
P_{loss} &= \frac{I}{e} U_s \\
&= \frac{(.5\ \mathrm{A})}{e}(.465\ \mathrm{GeV}) \\
&= .23\ \mathrm{GW}
\end{aligned}
$$

This means that the total RF power required would be about half a GigaWatt. Since RF cavities aren't particularly efficient, this means that such a machine would require need roughly a dedicated power plant just to compensate for synchrotron power loss.

5. *(8 pts)* An attempt is made to independently adjust the horizontal and vertical tunes in a 10 GeV (kinetic energy) proton synchrotron. It's found that the minimum tune separation that can be achieved is .02, due to coupling. Assume this is caused by a single, normal quadrupole which has been slightly rotated.

a. What is the value for the normalized coupling coefficient?

$$\kappa \equiv \sqrt{\beta_x \beta_y}\tilde{q} = \sqrt{\beta_x \beta_y}\frac{\tilde{B}' L}{(B\rho)}$$

(2 pts) In the coupled oscillation lecture, we found that for a machine with uncoupled tunes ν_x and ν_y, the presence of linear coupling would give rise to measured tunes

$$\nu_{1,2} = \bar{\nu} \pm \frac{1}{4\pi}\sqrt{\kappa^2 + 4\pi^2(\delta\nu)^2}$$

where $\bar{\nu} \equiv \frac{(\nu_x+\nu_y)}{2}$, $\delta\nu \equiv (\nu_y - \nu_x)$, and κ is the normalized coupling as described above. The minimum separation in the measured tunes will come when the nominal tunes are equal $\nu_x = \nu_y = \bar{\nu} \rightarrow\ \delta\nu = 0$, and the observed tunes will be

$$\nu_{1,2} = \bar{\nu} \pm \frac{\kappa}{4\pi}$$

which are separated by $\frac{\kappa}{2\pi}$. Therefore, the normalized coupling which would give rise to the minimum tune separation observed would be

$$
\begin{aligned}
\kappa &= 2\pi(\Delta\nu)_{\mathrm{min}} \\
&= 2\pi(.02) \\
&= .126
\end{aligned}
$$

b. If the quadrupole in question is 1 m long and has a gradient of 1 T/m, located at a point where β_x= 50m β_y=20m, what angular rotation (magnitude, in degrees) would cause this much coupling?

(6 pts) We can convert from the normalized coupling term above to the real skew sextupole term with

$$(\tilde{B}'L) = \kappa \frac{(B\rho)}{\sqrt{\beta_x \beta_y}}$$

We'll need to know the stiffness of the beam, so we calculate the momentum with

$$\begin{aligned} p &= \frac{1}{c}\sqrt{(K + mc^2)^2 - (mc^2)^2} \\ &= \sqrt{(10.938)^2 - (.938)^2} \\ &= 10.9 \frac{\mathrm{GeV}}{\mathrm{c}} \end{aligned}$$

$$\rightarrow (B\rho) = \frac{10.9}{.3} = 36.3\ \mathrm{T-m}$$

so the required integrated skew field will be

$$\begin{aligned} (\tilde{B}'L) &= \kappa \frac{(B\rho)}{\sqrt{\beta_x \beta_y}} \\ &= (.126)\frac{(36.3)}{\sqrt{(50)(20)}} \\ &= .144\ \mathrm{T} \end{aligned}$$

You proved in the homework, that a normal quadrupole which has been rotated by ϕ will have a skew term given by

$$\begin{aligned} \tilde{B}' &= B' \sin 2\phi \\ \rightarrow \tilde{B}'L &= B'L \sin 2\phi \end{aligned}$$

so the rotation needed to produce the skew field calculated above would be

$$\begin{aligned} \phi &= \frac{1}{2}\sin^{-1}\left(\frac{\tilde{B}'L}{B'L}\right) \\ &= \frac{1}{2}\sin^{-1}\left(\frac{.144}{(1)(1)}\right) \\ &= .072\ \mathrm{rad} \\ &= 4.1^\circ \end{aligned}$$