FITTING EMITTANCE AND σ_P/P. **2**

MOMENTUM DEVIATION CALCULATION USING RF FREQUENCY **3**
PHASE SLIP FACTOR η 3
$\delta P/P$ CALCULATION 3
DIFFERENTIAL RELATIONS BETWEEN B, R, $\Delta P/P$, AND F 3

LATTICE FUNCTION CALCULATION CLOSED ORBIT BPM DATA **4**
PHASE ADVANCE CALCULATION 4
BETA CALCULATION 4

RING CLOSURE CALCULATION **5**
TWISS PARAMETERS 5
DISPERSION FUNCTIONS 6
CLOSED ORBIT 7

FORMULA FOR CALCULATING CLOSED BUMPS **8**
3-BUMP FOR DESIRED DISPLACEMENT Δx 8
4-BUMP OF EQUAL DISPLACEMENT AT LOCATION 2 & 3 8
Angles for the 4-bump *9*
Displacements within the 4-bump at any K location *9*

CALCULATION OF MULTI-MOMENTS, SIGMA, SKEWNESS AND KURTOSIS **10**
DISCRETE N-TH MOMENT CALCULATION 10
SIGMA 10
SKEWNESS 10
KURTOSIS 10

ROTATION OF ELLIPSES AND FORMULA **11**
ELLIPSE IN POLAR COORDINATE 11
ELLIPSE IN PARAMETRIC FORMULA, USING LATTICE PARAMETERS 11
ROTATE ELLIPSE TO ZERO TILT 11
ROTATING AN ELLIPSE 12

FITTING PROFILE SIGMA FOR EMITTANCE AND $\Delta P/P$ **15**
FORMULA 15
FITTING FOR BOTH EMITTANCE AND $\Delta P/P$ 15
FITTING ONLY FOR EMITTANCE 16

BASIC COORDINATE ROTATION **17**
ROTATE BY AN ANGLE θ 17

FITTING PERIODIC PATTERN WITH SINE FUNCTION **18**
PHASE AND AMPLITUDE FITTING 18
ERROR OF FIT 20
SCANNING FOR THE BEST FREQUENCY 20

FINDING MINIMUM OF A 3RD ORDER POLYNOMIAL FUNCTION **21**
MOTIVATION 21
FORMULATION 21

CALCULATING EXACT QUADRATIC EQUATION FROM 3 POINTS **22**
APPLICATION 22
FORMULATION 22

Fitting emittance and σ_p/p.

$$S = \sum_i^N w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]^2$$

$$\frac{\partial S}{\partial \varepsilon} = \sum_i w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]\beta_i = 0$$

$$\frac{\partial S}{\partial \delta^2} = \sum_i w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]\eta_i^2 = 0$$

$$\begin{pmatrix} \sum_i w_i\sigma_i^2\beta_i \\ \sum_i w_i\sigma_i^2\eta_i^2 \end{pmatrix} = \begin{bmatrix} \sum_i w_i\beta_i^2 & \sum_i w_i\eta_i^2\beta_i \\ \sum_i w_i\eta_i^2\beta_i & \sum_i w_i\eta_i^4 \end{bmatrix} \cdot \begin{pmatrix} \varepsilon \\ \delta^2 \end{pmatrix}$$

$$S = \sum_i w_i[\sigma_i^4 - 2\varepsilon\sigma_i^2\beta_i - 2\delta^2\sigma_i^2\eta_i^2 + \varepsilon^2\beta_i^2 + \delta^4\eta_i^4 + 2\varepsilon\delta^2\beta_i\eta_i^2]$$

$$S = \sum_i w_i[\sigma_i^4 - 2\varepsilon\sigma_i^2\beta_i - 2\delta^2\sigma_i^2\eta_i^2 + \varepsilon^2\beta_i^2 + \delta^4\eta_i^4 + 2\varepsilon\delta^2\beta_i\eta_i^2]$$

$$= \sum_i w_i\sigma_i^4 - 2\varepsilon\sum_i w_i\sigma_i^2\beta_i - 2\delta^2\sum_i w_i\sigma_i^2\eta_i^2 + \varepsilon^2\sum_i w_i\beta_i^2 + \delta^4\sum_i w_i\eta_i^4 + 2\varepsilon\delta^2\sum_i w_i\beta_i\eta_i^2$$

$$\chi^2 = S/(N-2)$$

$$\begin{pmatrix} \varepsilon \\ \delta^2 \end{pmatrix} = \Re \cdot \begin{pmatrix} \sum_i w_i\sigma_i^2\beta_i \\ \sum_i w_i\sigma_i^2\eta_i^2 \end{pmatrix}$$

$$\Delta\varepsilon = \sqrt{\chi^2 \cdot \Re_{11}}$$

$$\Delta\delta^2 = \sqrt{\chi^2 \cdot \Re_{22}}$$

$$\Delta\delta = \frac{\sqrt{\chi^2 \cdot \Re_{22}}}{2\delta}$$

$$\Delta\delta = \sqrt{\chi^2 \cdot \Re_{22}} \Big/ 2\delta$$

Momentum deviation calculation using RF frequency

Phase slip factor η

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

The transition gamma γ_t is 5.401 for Booster and 21.6 for Main Injector.

δp/p calculation

$$\frac{\Delta p}{p} = -\frac{1}{\eta} \cdot \frac{\Delta f}{f}$$

Differential relations between *B*, *R*, Δ*p*/*p*, and *f*

Variables	**Equations**
B, p, R	$\frac{\Delta p}{p} = \gamma_{tr}^2 \cdot \frac{\Delta R}{R} + \frac{\Delta B}{B}$
f, p, R	$\frac{\Delta p}{p} = \gamma^2 \cdot \frac{\Delta f}{f} + \gamma^2 \cdot \frac{\Delta R}{R}$
B, f, p	$\frac{\Delta B}{B} = \gamma_{tr}^2 \cdot \frac{\Delta f}{f} + \frac{\gamma^2 - \gamma_{tr}^2}{\gamma^2} \cdot \frac{\Delta p}{p}$
B, f, R	$\frac{\Delta B}{B} = \gamma^2 \cdot \frac{\Delta f}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \cdot \frac{\Delta R}{R}$

Lattice function calculation closed orbit BPM data

Phase advance calculation

The 1-bump orbit displacement at i-th BPM caused by the dipole correctors can be written as:

$$x_i^{(1)} = \frac{\sqrt{\beta_i \beta_1} \cdot k_1 \cos\left[2\pi(\psi_i - \alpha_1)\right]}{2\sin(\pi Q)}, \quad \text{with } \alpha_1 = \psi_1 + Q/2 \text{ and} \tag{1a}$$

$$x_i^{(2)} = \frac{\sqrt{\beta_i \beta_2} \cdot k_2 \cos\left[2\pi(\psi_i - \alpha_2)\right]}{2\sin(\pi Q)}, \quad \text{with } \alpha_2 = \psi_2 + Q/2 \text{ and} \tag{1b}$$

Q is the total tune of the machine. The indices 1 & 2 are used to denote the two dipole corrector used to cause 1-bump orbit, i.e. ψ_1 and ψ_2 are the phase at the corrector location. The indices i denote the i-th BPM location.

Define $a_i^{(1)} = \frac{2\sin(\pi Q)}{\sqrt{\beta_1} \cdot k_1} x_i^{(1)}$ and $a_i^{(2)} = \frac{2\sin(\pi Q)}{\sqrt{\beta_2} \cdot k_2} x_i^{(2)}$ and we get:

$$\cos(2\pi\psi_i) = \frac{1}{\sqrt{\beta_i}} \frac{a_i^{(1)} \sin(2\pi\alpha_2) - a_i^{(2)} \sin(2\pi\alpha_1)}{\sin\left[2\pi(\alpha_2 - \alpha_1)\right]} \tag{2a}$$

$$\sin(2\pi\psi_i) = \frac{1}{\sqrt{\beta_i}} \frac{a_i^{(1)} \cos(2\pi\alpha_2) - a_i^{(2)} \cos(2\pi\alpha_1)}{-\sin\left[2\pi(\alpha_2 - \alpha_1)\right]} \tag{2b}$$

From equation (2) we could calculate ψ_i without knowing what the value of β_i is.

beta calculation

Ring closure calculation

Twiss parameters

Treating a ring as transfer line and start with initial twiss parameter of β_1 and α_1, we get at one turn around β_2, α_2, and phase advance ψ_{12}. The 1-turn transfer matrix is:

$$M = \begin{bmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) & \left(\beta_1\beta_2\right)^{\frac{1}{2}} \sin\psi_{12} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{12} - \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\psi_{12} & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right) \end{bmatrix} \tag{1}$$

This same matrix can be expressed in the actual ring twiss parameters β, α, and total phase advance ψ as:

$$M = \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \tag{2}$$

From equation (1) and (2) we get:

$$\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) = \cos\psi + \alpha\sin\psi \tag{3a}$$

$$\left(\beta_1\beta_2\right)^{\frac{1}{2}} \sin\psi_{12} = \beta\sin\psi \tag{3b}$$

$$-\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{12} - \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\psi_{12} = -\frac{1+\alpha^2}{\beta}\sin\psi \tag{3c}$$

$$\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right) = \cos\psi - \alpha\sin\psi \tag{3d}$$

Using equ. (3a) and (3d) we get:

$$\begin{aligned} \cos\psi &= 0.5 \cdot \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right)\right] \\ &= 0.5 \cdot \left\{\left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\right] \cdot \cos\psi_{12} + \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\alpha_1 - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\alpha_2\right] \cdot \sin\psi_{12}\right\} \end{aligned} \tag{4a}$$

and

$$\sin\psi = \sqrt{1-\cos^2\psi} \tag{4b}$$

With equation (3b) we have:

$$\beta = \frac{(\beta_1\beta_2)^{\frac{1}{2}} \sin\psi_{12}}{\sin\psi} \tag{5}$$

with the sign of $\sin\psi$ being such that β is positive.

By subtracting equ. (3d) from (3a) the value of α follows:

$$\alpha = \frac{\left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\right] \cdot \cos\psi_{12} + \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} \alpha_1 + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}} \alpha_2\right] \cdot \sin\psi_{12}}{2\sin\psi} \tag{6}$$

Dispersion functions

In similar approach we treat the ring as beamline and track the dispersion function all the way around with η_1, $\eta_1^{'}$, being the initial condition and η_2, $\eta_2^{'}$ being the result at 1-turn. To find the right initial condition such that the initial and final condition are the same we can write down the equation as:

$$\begin{bmatrix} \eta_2 \\ \eta_2^{'} \end{bmatrix} + \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \begin{pmatrix} \Delta\eta \\ \Delta\eta^{'} \end{pmatrix} = \begin{pmatrix} \eta_1 + \Delta\eta \\ \eta_1^{'} + \Delta\eta^{'} \end{pmatrix} = \begin{pmatrix} \eta \\ \eta^{'} \end{pmatrix}$$

where η and $\eta^{'}$ are the closed solution. Substituting $\Delta\eta$ with $\eta - \eta_1$ and $\Delta\eta^{'}$ with $\eta^{'} - \eta_1^{'}$ we get:

$$\begin{bmatrix} \eta_2 \\ \eta_2^{'} \end{bmatrix} + \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \begin{pmatrix} \eta - \eta_1 \\ \eta^{'} - \eta_1^{'} \end{pmatrix} = \begin{pmatrix} \eta \\ \eta^{'} \end{pmatrix}$$

This gives us two equations from with to solve for η and $\eta^{'}$:

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{pmatrix} \eta \\ \eta^{'} \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

where

$$a_0 = \eta_2 - \left[(\cos\psi + \alpha\sin\psi) \cdot \eta_1 + \beta\sin\psi \cdot \eta_1^{'}\right]$$

$$a_1 = 1 - \cos\psi - \alpha\sin\psi$$

$$a_2 = -\beta\sin\psi$$

$$b_0 = \eta_2^{'} - \left[\frac{-(1+\alpha^2)}{\beta}\sin\psi \cdot \eta_1 + (\cos\psi - \alpha\sin\psi) \cdot \eta_1^{'}\right]$$

$$b_1 = \frac{1+\alpha^2}{\beta}\sin\psi$$

$$b_2 = 1 - \cos\psi + \alpha\sin\psi$$

Closed orbit

The equation to solve for the closed orbit is exactly the same as that of the dispersion function, except that η is replaced by x.

$$\begin{bmatrix} x_2 \\ x_2^{'} \end{bmatrix} + \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\dfrac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \begin{pmatrix} x - x_1 \\ x^{'} - x_1^{'} \end{pmatrix} = \begin{pmatrix} x \\ x^{'} \end{pmatrix}$$

Among the coefficients above only a_0 and b_0 need to be recalculated. The initial condition for the horizontal plane and vertical plane closed orbit can be solved the same way.

Formula for calculating closed bumps

3-Bump for desired displacement Δx

$$\theta_1 = \frac{\Delta x}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\theta_2 = -\theta_1\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\theta_3 = \theta_1\left(\frac{\beta_1}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

4-Bump of equal displacement at location 2 & 3

Use the equations above we can construct two adjacent 3-bumps. The first is:

$$\alpha_1 = \frac{\Delta x_2}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\alpha_2 = -\alpha_1\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\alpha_3 = \alpha_1\left(\frac{\beta_1}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

By making $\Delta x_3 = \Delta x_2 = \alpha_1\sqrt{\beta_1\beta_2}\,\sin\psi_{12}$ and with final 4-bump angles as: $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2 + \gamma_2$, $\theta_3 = \alpha_3 + \gamma_3$, and $\theta_4 = \alpha_4$, the second 3-bump can be written as:

$$\gamma_2 = \frac{\Delta x_3}{\sqrt{\beta_2\beta_3}\,\sin\psi_{23}} = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

$$\gamma_3 = -\gamma_2\left(\frac{\beta_2}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{24}}{\sin\psi_{34}} = -\theta_1\frac{\sqrt{\beta_1\beta_2}}{\beta_3}\,\frac{\sin\psi_{12}\cdot\sin\psi_{24}}{\sin\psi_{23}\cdot\sin\psi_{34}}\ ,$$

$$\gamma_4 = \gamma_2\left(\frac{\beta_2}{\beta_4}\right)^{\frac{1}{2}}\frac{\sin\psi_{23}}{\sin\psi_{34}} = \theta_1\sqrt{\frac{\beta_1\beta_2}{\beta_3\beta_4}}\,\frac{\sin\psi_{12}}{\sin\psi_{34}}$$

Angles for the 4-bump

$$\theta_1 = \alpha_1 = \frac{\Delta x_2}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\theta_2 = \alpha_2 + \gamma_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\,\frac{\sin\psi_{13}}{\sin\psi_{23}} + \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}} = \frac{-\theta_1\sqrt{\beta_1}}{\sin\psi_{23}}\left(\frac{\sin\psi_{13}}{\sqrt{\beta_2}} - \frac{\sin\psi_{12}}{\sqrt{\beta_3}}\right)$$

$$\theta_3 = \alpha_3 + \gamma_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}} - \theta_1\frac{\sqrt{\beta_1\beta_2}}{\beta_3}\,\frac{\sin\psi_{12}\cdot\sin\psi_{24}}{\sin\psi_{23}\cdot\sin\psi_{34}} = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}\left(1 - \sqrt{\frac{\beta_2}{\beta_3}}\,\frac{\sin\psi_{24}}{\sin\psi_{34}}\right)$$

$$\theta_4 = \gamma_4 = \theta_1\frac{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}{\sqrt{\beta_3\beta_4}\,\sin\psi_{34}}$$

Displacements within the 4-bump at any K location

$\Delta x_K = \sum_{i=1,2,3,4}\theta_i\sqrt{\beta_i\beta_K}\,\sin\psi_{iK}$, for all i's with $\psi_{iK} > 0$.

Calculation of multi-moments, sigma, skewness and kurtosis

Discrete n-th moment calculation

Using the cumulant, or raw moments

$$s_n = \sum_i x_i^n y_i$$

and the mean

$$\bar{x} = \frac{1}{s_0} \sum_i x_i y_i = \frac{s_1}{s_0}$$

The n-th moment $\mu_n = \frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^n \cdot y_i$ can be calculated.

Sigma

$$\sigma^2 = \mu_2 = \frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^2 \cdot y_i = \frac{1}{s_0} \sum_i x_i^2 y_i - \bar{x}^2 = \frac{s_2}{s_0} - \bar{x}^2$$

skewness

$$\zeta = \frac{\mu_3}{\sigma^3} = \frac{1}{\sigma^3} \left[\frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^3 \cdot y_i \right] = \frac{1}{\sigma^3} \left[\frac{s_3}{s_0} - 3\bar{x} \cdot \frac{s_2}{s_0} + 2\bar{x}^3 \right]$$

Kurtosis

$$\kappa = \frac{\mu_4}{\sigma^4} = \frac{1}{\sigma^4} \left[\frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^4 \cdot y_i \right] = \frac{1}{\sigma^4} \left[\frac{s_4}{s_0} - 4\bar{x} \cdot \frac{s_3}{s_0} + 6\bar{x}^2 \cdot \frac{s_2}{s_0} - 3\bar{x}^4 \right]$$

and the ***excess kurtosis***

$\kappa_{ex} = \kappa - 3.$

Rotation of ellipses and formula

Ellipse in polar coordinate

The equation for an un-rotated ellipse, i.e. major axis on the x-axis, can be written down as $r = \frac{a^2 - f^2}{a - f \cdot \cos\theta}$, with a being the half length of major axis and f being the distance from center of ellipse to the focal point. It is also conventionally written as $r = \frac{p}{1 - e \cdot \cos\theta}$.

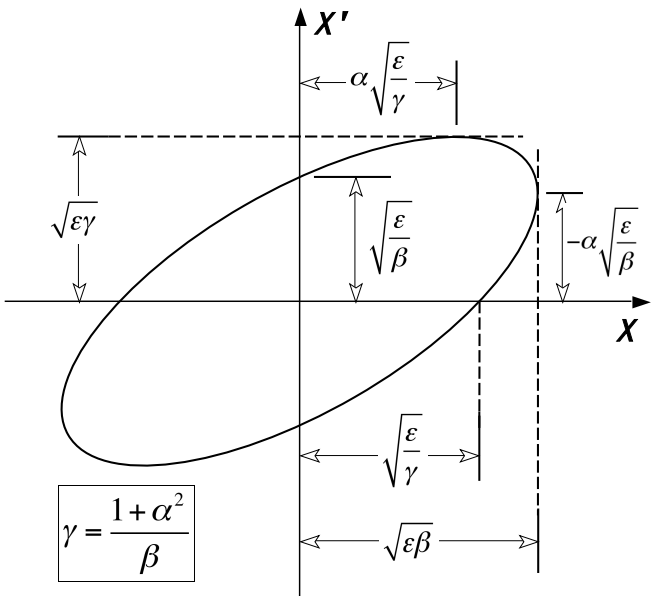


Figure 1: Characteristics of an elliptical contour.

Ellipse in parametric formula, using lattice parameters

$$\begin{cases} x = \sqrt{\varepsilon\beta}\cos\phi \\ x' = -\sqrt{\frac{\varepsilon}{\beta}}(\sin\phi + \alpha \cdot \cos\phi) \end{cases} \tag{1}$$

Where β and α are lattice parameters, and ε the emittance.

Rotate ellipse to zero tilt

To un-tilt a rotated ellipse the angle of the major axis need to be determined. This can be accomplished by utilizing the fact that

$$\vec{r}_A \bullet \Delta(\vec{r}_A) = 0, \tag{2}$$

Where $\vec{r}_A = \begin{bmatrix} x_A \\ x'_A \end{bmatrix}$, using expression given in (1), is the vector from origin of ellipse to point A, where major axis intersects the ellipse. The other vector $\Delta(\vec{r}_A) = \begin{bmatrix} \frac{dx}{d\phi} \\ \frac{dx'}{d\phi} \end{bmatrix}_A$ is the derivative of $\vec{r}_A$ and is given as

$$\begin{cases} \frac{dx}{d\phi} = -\sqrt{\varepsilon\beta}\sin\phi_A \\ \frac{dx'}{d\phi} = -\sqrt{\frac{\varepsilon}{\beta}}\left(\cos\phi_A - \alpha\sin\phi_A\right) \end{cases}, \tag{3}$$

with ϕ_A being the phase at point A.
Applying expression (1) and (3) in the dot-product equation (2) leads to

$$\tan\left(2\phi_A\right) = \frac{2\alpha}{\beta^2 + \alpha^2 - 1},$$

The direction of the major axis can now be calculated as:

$$\theta = \tan^{-1}\left(\frac{x'_A}{x_A}\right) = \tan^{-1}\left(\frac{-\sqrt{\frac{\varepsilon}{\beta}}(\sin\phi_A + \alpha\cdot\cos\phi_A)}{\sqrt{\varepsilon\beta}\cos\phi_A}\right) = \tan^{-1}\left(\frac{-\left(\sin\phi_A + \alpha\cdot\cos\phi_A\right)}{\beta\cdot\cos\phi_A}\right),$$

and the half-length of major axis a as:

$$a = \sqrt{\left(\sqrt{\varepsilon\beta}\cos\phi_A\right)^2 + \left(-\sqrt{\frac{\varepsilon}{\beta}}(\sin\phi_A + \alpha\cdot\cos\phi_A)\right)^2}$$

The new beta function β_0 of the un-tilted ellipse, call it a ***normal ellipse*** for now, can be written as:

$$\beta_0 = \frac{a^2}{\varepsilon} = \beta\cos^2\phi_A + \frac{1}{\beta}\left(\sin\phi_A + \alpha\cdot\cos\phi_A\right)^2,$$

with $\alpha_0 = 0$ by default.

Rotating an ellipse

An ellipse is typically represented by:

$$\begin{cases} x = \sqrt{\varepsilon\beta}\cos\phi \\ x' = -\sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right) \end{cases}$$

Standard rotational transformation can be used to rotate the ellipse by an angle θ:

$$\begin{bmatrix} x_\theta(\phi) \\ x_\theta'(\phi) \end{bmatrix} = \begin{bmatrix} x\cos\theta + x'\sin\theta \\ -x\sin\theta + x'\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\varepsilon\beta}\cos\phi\cos\theta - \sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\sin\theta \\ -\sqrt{\varepsilon\beta}\cos\phi\sin\theta - \sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\cos\theta \end{bmatrix}.$$

The point on the ellipse with $x_\theta = x_{Max} = \sqrt{\varepsilon\beta_\theta}$ can be found by taking the derivative of $x_\theta(\phi)$ with respect to ϕ,

$$\frac{dx_\theta(\phi)}{d\phi} = \frac{d}{d\phi}\left(\sqrt{\varepsilon\beta}\cos\phi\cos\theta + \left(-\sqrt{\frac{\varepsilon}{\beta}}\right)\left(\sin\phi + \alpha\cos\phi\right)\cdot\sin\theta\right)$$

$$= -\sqrt{\varepsilon\beta}\sin\phi\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi\sin\theta.$$

The condition that $\left[\dfrac{dx_\theta(\phi)}{d\phi}\right]_{\phi=\phi_{X\max}} = 0$ leads to:

$$\left[\frac{dx_\theta(\phi)}{d\phi}\right]_{\phi=\phi_{X\max}} = -\sqrt{\varepsilon\beta}\sin\phi_{X\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X\max}\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi_{X\max}\sin\theta$$

$$= \sin\phi_{X\max}\cdot\left(-\sqrt{\varepsilon\beta}\cos\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\theta\right) - \cos\phi_{X\max}\cdot\left(\sqrt{\frac{\varepsilon}{\beta}}\sin\theta\right)$$

$$= 0$$

and

$$\phi_{X\max} = \tan^{-1}\left(\frac{-\tan\theta}{\beta - \alpha\tan\theta}\right).$$

Using the characteristics shown in Figure 1 the new β_θ and α_θ for the rotated ellipse can be calculated:

$$x_\theta(\phi_{X\max}) = \sqrt{\varepsilon\beta}\cos\phi_{X\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X\max} + \alpha\cos\phi_{X\max}\right)\cdot\sin\theta$$

$$= \sqrt{\varepsilon\beta_\theta}$$

$$x_\theta'(\phi_{X\max}) = -\sqrt{\varepsilon\beta}\cos\phi_{X\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X\max} + \alpha\cos\phi_{X\max}\right)\cdot\cos\theta$$

$$= -\alpha_\theta \cdot \sqrt{\frac{\varepsilon}{\beta_\theta}}$$

Similarly setting the derivative of $x_\theta'(\phi)$ to zero,

$$\left.\frac{d\left[x_\theta'(\phi)\right]}{d\phi}\right|_{\phi=\phi_{X'\max}} = \left.\frac{d}{d\phi}\left(-\sqrt{\varepsilon\beta}\cos\phi\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\cos\theta\right)\right|_{\phi=\phi_{X'\max}}$$

$$= \sqrt{\varepsilon\beta}\sin\phi_{X'\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X'\max}\cos\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi_{X'\max}\cos\theta$$

$$= \sin\phi_{X'\max}\left(\sqrt{\varepsilon\beta}\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\cos\theta\right) - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X'\max}\cos\theta$$

$$= 0$$

will then lead to

$$\phi_{X'\max} = \tan^{-1}\left(\frac{1}{\beta\cdot\tan\theta + \alpha}\right) \quad .$$

Plugging in $\phi_{X'\max}$ to get

$$x_\theta'(\phi_{X'\max}) = -\sqrt{\varepsilon\beta}\cos\phi_{X'\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X'\max} + \alpha\cos\phi_{X'\max}\right)\cdot\cos\theta$$

$$= \sqrt{\varepsilon\gamma_\theta} = \sqrt{\varepsilon\cdot\frac{1+\alpha_\theta^2}{\beta_\theta}}$$

and

$$x_\theta(\phi_{X'\max}) = \sqrt{\varepsilon\beta}\cos\phi_{X'\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X'\max} + \alpha\cos\phi_{X'\max}\right)\cdot\sin\theta$$

$$= \alpha_\theta\sqrt{\frac{\varepsilon}{\gamma_\theta}} = \alpha_\theta\sqrt{\frac{\varepsilon\beta_\theta}{1+\alpha_\theta^2}} \quad .$$

new lattice function β_θ and α_θ can be calculated accordingly:

$$\alpha_\theta = \frac{x_\theta(\phi_{X'\max})\cdot x'_\theta(\phi_{X'\max})}{\varepsilon}$$

$$\gamma_\theta = \frac{\left[x'_\theta(\phi_{X'\max})\right]^2}{\varepsilon}$$

$$\beta_\theta = \frac{1+\alpha_\theta^2}{\gamma_\theta}$$

Fitting profile sigma for emittance and Δp/p

Formula

The observed beam profile sigma σ_i at the *i*-th profile monitor can be written down as

$$\sigma_i^2 = \varepsilon \cdot \beta_i + \eta_i^2 \cdot \delta^2$$

with β_i and η_i being the corresponding beta and dispersion function. The emittance and Δp/p, i.e. ε and δ, can be found by minimizing the action sum:

$$S = \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 .$$

Taking the derivative of *S* with respective to ε we get:

$$\begin{aligned}
\frac{\partial S}{\partial \varepsilon} &= \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 \\
&= \sum_i -2\beta_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right) \\
&= \sum_i -2\beta_i \sigma_i^2 + \sum_i -2\beta_i \left(-\varepsilon \cdot \beta_i\right) + \sum_i -2\beta_i \left(-\eta_i^2 \cdot \delta^2\right) = 0 \\
\Rightarrow \varepsilon \cdot \sum_i \beta_i^2 + \delta^2 \cdot \sum_i \beta_i \eta_i^2 &= \sum_i \beta_i \sigma_i^2
\end{aligned} \tag{1}$$

Taking the derivative of *S* with respect to we get:

$$\begin{aligned}
\frac{\partial S}{\partial \delta^2} &= \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 \\
&= \sum_i -2\eta_i^2 \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right) \\
&= \sum_i -2\eta_i^2 \sigma_i^2 + \sum_i -2\eta_i^2 \left(-\varepsilon \cdot \beta_i\right) + \sum_i -2\eta_i^2 \left(-\eta_i^2 \cdot \delta^2\right) = 0 \\
\Rightarrow \varepsilon \cdot \sum_i \eta_i^2 \beta_i + \delta^2 \cdot \sum_i \eta_i^4 &= \sum_i \eta_i^2 \sigma_i^2
\end{aligned} \tag{2}$$

Fitting for both emittance and Δp/p

From (1) and (2) a matrix equation can be constructed as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \delta^2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3}$$

where

$$a_{11} = \sum_i \beta_i^2 \ , \ a_{12} = \sum_i \beta_i \eta_i^2 \ ,$$

$$a_{21} = \sum_i \eta_i^2 \beta_i \,,\; a_{22} = \sum_i \eta_i^4 \,,$$

and

$$V_1 = \sum_i \beta_i \sigma_i^2 \,,\; V_2 = \sum_i \eta_i^2 \sigma_i^2$$

The values for ε and δ can now be found by solving the 2 × 2 equations (3).

Fitting only for emittance

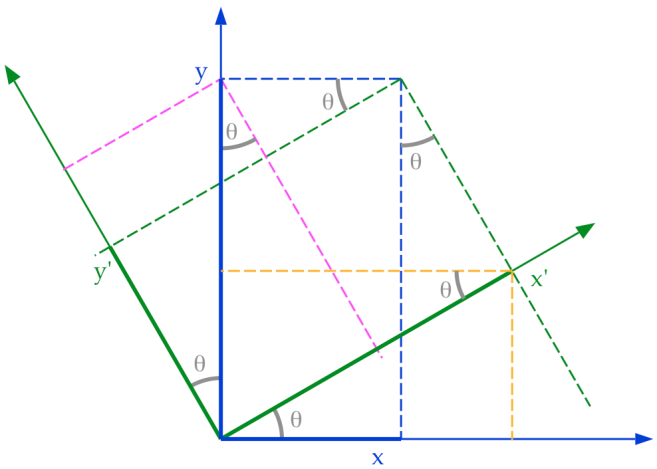
In this case δ is assumed known and ε can be calculated readily:

$$\varepsilon = \frac{V_1 - \delta^2 \cdot a_{12}}{a_{11}} \text{ , or}$$

$$\varepsilon = \frac{V_2 - \delta^2 \cdot a_{22}}{a_{21}}$$

Basic coordinate rotation

<u>Rotate by an angle **θ**</u>



From the drawing above we can write down the transformation matrix as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and the inverse of it:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Fitting periodic pattern with sine function

Phase and amplitude fitting

For a given oscillation frequency Ω the best matching φ, amplitude A, and overall offset δ can be determine through least square minimization.
The functional value of the oscillation can be written as:

$$\begin{aligned} f(t_i) &= A \cdot e^{-\frac{t_i}{\tau}} \cdot \sin(\Omega \cdot t_i + \varphi) + \delta \\ &= A \cdot e^{-\frac{t_i}{\tau}} \cdot \sin(\Omega \cdot t_i) \cdot \cos\varphi + A \cdot e^{-\frac{t_i}{\tau}} \cdot \cos(\Omega \cdot t_i) \cdot \sin\varphi + \delta \\ &= a \cdot e^{-\frac{t_i}{\tau}} \cdot \sin(\Omega \cdot t_i) + b \cdot e^{-\frac{t_i}{\tau}} \cdot \cos(\Omega \cdot t_i) + \delta \\ &= a \cdot S_i + b \cdot C_i + \delta \end{aligned},$$

Where a, b, S_i, and C_i are defined as:

$$\begin{aligned} a &= A\cos\varphi \\ b &= A\sin\varphi \\ S_i &= e^{-\frac{t_i}{\tau}} \cdot \sin(\Omega \cdot t_i) \\ C_i &= e^{-\frac{t_i}{\tau}} \cdot \cos(\Omega \cdot t_i) \end{aligned}.$$

The action S, the deviation square sum, can be written as:

$$\begin{aligned} S &= \sum_i \left[y_i - f(t_i) \right]^2 \\ &= \sum_i \left[y_i - a \cdot S_i - b \cdot C_i - \delta \right]^2 \end{aligned}$$

Minimizing S with respect to the three variables gives:

$$\begin{aligned} \frac{\partial S}{\partial a} &= \sum_i -S_i \left[y_i - a \cdot S_i - b \cdot C_i - \delta \right] \\ &= -\sum_i y_i S_i + a \cdot \sum_i S_i^2 + b \cdot \sum_i S_i \cdot C_i + \delta \cdot \sum_i S_i = 0 \\ \frac{\partial S}{\partial b} &= \sum_i -C_i \left[y_i - a \cdot S_i - b \cdot C_i - \delta \right] \\ &= -\sum_i y_i C_i + a \cdot \sum_i S_i C_i + b \cdot \sum_i C_i^2 + \delta \cdot \sum_i C_i = 0 \\ \frac{\partial S}{\partial \delta} &= \sum_i -\left[y_i - a \cdot S_i - b \cdot C_i - \delta \right] \\ &= -\sum_i y_i + a \cdot \sum_i S_i + b \cdot \sum_i C_i + \delta \cdot N = 0 \end{aligned}$$

This leads to matrix equation for a, b, and δ as:

$$\mathbf{X} \cdot \begin{bmatrix} a \\ b \\ \delta \end{bmatrix} = Y$$

where

$$\mathbf{X} = \begin{bmatrix} \sum_i S_i^2 & \sum_i S_i C_i & \sum_i S_i \\ \sum_i S_i C_i & \sum_i C_i^2 & \sum_i C_i \\ \sum_i S_i & \sum_i C_i & N \end{bmatrix},$$

and

$$Y = \begin{bmatrix} \sum_i y_i S_i \\ \sum_i y_i C_i \\ \sum_i y_i \end{bmatrix}$$

The solution then becomes:

$$\begin{bmatrix} a \\ b \\ \delta \end{bmatrix} = \mathbf{X}^{-1} \cdot Y \, .$$

and

$$A = \sqrt{a^2 + b^2}$$

$$\varphi = \tan^{-1}\left(\frac{b}{a}\right)$$

Error of fit

Action S can now be evaluated readily with the newly determined a, b, and δ:

$$
\begin{aligned}
S &= \sum_i \left[y_i - a \cdot S_i - b \cdot C_i - \delta \right]^2 \\
&= \sum_i y_i^2 + a^2 \cdot \sum_i S_i^2 + b^2 \cdot \sum_i C_i^2 + \delta^2 N \\
&\quad -2a \cdot \sum_i y_i \cdot S_i - 2b \cdot \sum_i y_i \cdot C_i - 2\delta \cdot \sum_i y_i \\
&\quad +2ab \cdot \sum_i S_i C_i + 2a\delta \cdot \sum_i S_i + 2b\delta \cdot \sum_i C_i \\
&= \left(\sum_i y_i^2 \right) + a^2 \cdot X_{11} + b^2 \cdot X_{22} + \delta^2 \cdot X_{33} \\
&\quad -2a \cdot Y_1 - 2b \cdot Y_2 - 2\delta \cdot Y_3 + 2ab \cdot X_{21} + 2a\delta \cdot X_{31} + 2b\delta \cdot X_{32}
\end{aligned}
$$

Scanning for the best frequency

Though best matching frequency can't be calculated through minimization one can still find its value by using iterative scanning procedure. The value of action S as a function of Ω can be used to determine the best matching frequency. A restricted frequency range should be used to avoid possible aliasing effect.

Finding minimum of a 3rd order polynomial function

Motivation

The minimum of a distribution can usually be found by using a 2nd order polynomial fit to the data. However, when the distribution around the minimum is not symmetric a 3rd order fit can provide more accuracy to the location of the minimum.

Formulation

Given a 3rd order polynomial

$$y = a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$$

The condition for either a local maximum or a local minimum is:

$$y' = 3a_3 \cdot x^2 + 2a_2 \cdot x + a_1 = 0$$

This leads to:

$$x_{\pm} = \frac{-a_2 \pm \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3}$$

and

$$\begin{aligned} y_{\pm}'' &= 6a_3 \cdot x_{\pm} + 2a_2 \\ &= 6a_3 \cdot \frac{-a_2 \pm \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3} + 2a_2 \\ &= \pm 2\sqrt{a_2^2 - 3a_3 \cdot a_1} \end{aligned}$$

Since the condition for a local minimum is that $y'' > 0$ the minimum is always with:

$$x_{+} = \frac{-a_2 + \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3}$$

The ordering of minimum and maximum will depend on the sign of a_3, i.e. the minimum will be on the negative side of maximum if $a_3 < 0$.

Calculating exact quadratic equation from 3 points

Application

For interpolating or extrapolating using existing data points.

Formulation

With

$$y_i = ax_i^2 + bx_i + c \text{, where } i = 1, 2, \text{ or } 3,$$

we can write down three difference equations:

$$y_2 - y_1 = a \cdot \left(x_2^2 - x_1^2\right) + b \cdot \left(x_2 - x_1\right)$$

$$y_3 - y_2 = a \cdot \left(x_3^2 - x_2^2\right) + b \cdot \left(x_3 - x_2\right),$$

$$y_3 - y_1 = a \cdot \left(x_3^2 - x_1^2\right) + b \cdot \left(x_3 - x_1\right)$$

and will need only two of them.

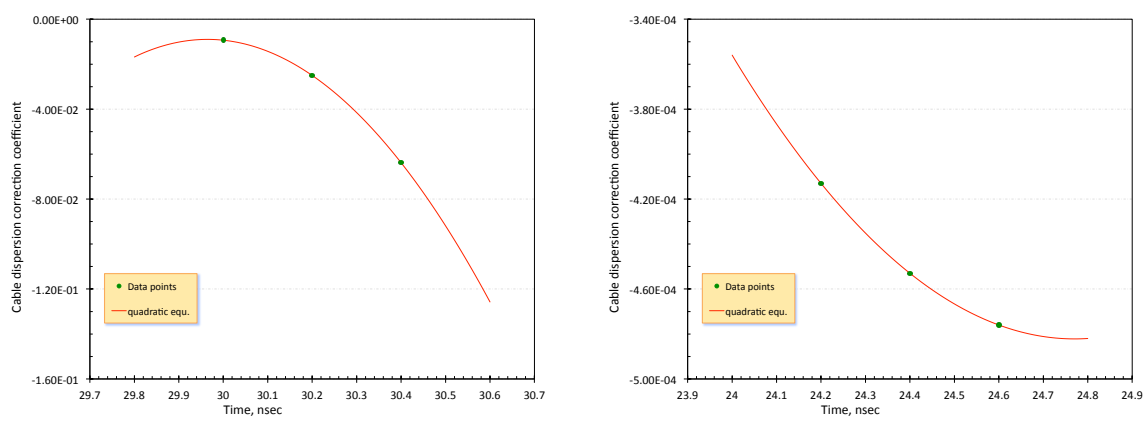
Using first two equations the solution for coefficients ***a*** and ***b*** can be written as:

$$a = \frac{\left(y_2 - y_1\right)\left(x_3 - x_2\right) - \left(y_3 - y_2\right)\left(x_2 - x_1\right)}{\left(x_2^2 - x_1^2\right)\left(x_3 - x_2\right) - \left(x_3^2 - x_2^2\right)\left(x_2 - x_1\right)}$$

and

$$b = -\frac{\left(y_2 - y_1\right)\left(x_3^2 - x_2^2\right) - \left(y_3 - y_2\right)\left(x_2^2 - x_1^2\right)}{\left(x_2^2 - x_1^2\right)\left(x_3 - x_2\right) - \left(x_3^2 - x_2^2\right)\left(x_2 - x_1\right)}.$$

The third constant ***c*** can be calculated when values for ***a*** and ***b*** have been determined.



Two data examples with their corresponding quadratic equations as calculated.