XIII. Classic Problems with Cosmology and Inflation

A major virtue of the Big Bang theory is its simplicity: given the fact that the universe is expanding, one is able to derive the equations governing the expansion with very little freedom in the details of the theory. However, the theory is hardly complete: there is no explanation of how or why the Big Bang began in the first place. Furthermore, the observed universe is hardly smooth on very small scales, and there is no explanation of how structure formed or what its amplitude should be. Rather than attacking the origin question head-on (it is still an unsolved question), the most sensible approach is to extrapolate the observed universe back in time and see what properties of the universe at early times manifest themselves today.

In the context of classical cosomology there are two paradoxes (to be explained below) that appear to place extremely demanding constraints on the big bang. These paradoxes are known as the flatness and horizon problems. The resolution of these paradoxes may well be found, not in the big bang itself, but rather in the physics of the universe at subsequent times.

The physics of the early universe is a fascinating and still developing field. It will largely be ignored here, mainly to keep the scope of the book limited. Extrapolating backwards in time beginning with the radiation-dominated phase of the universe, one finds that at early times the evolution of the universe depends as much on processes involving microscopic physics (atomic, nuclear, particle) as it does on macroscopic (gravitational) physics. Because the density and temperature of the universe increase without limit as one extrapolates back in time, one quickly finds that there is an epoch beyond which our current physical understanding is incomplete. (Roughly speaking, "known physics" referes to the physics of temperatures that correspond to energies that can be reproduced in present day particle accelerators). At present this epoch occurs at roughly $t = 10^{-??}$ seconds, when the universe has a temperature $T = 10^{10}$ K.

A. Flatness problem

This problem has already been alluded to. In an expanding universe, the parameter Ω ($= \rho/\rho_{crit}$) is not constant with time. Rather, if it deviates from 1 (either high or low), then that deviation grows with time. Conversely, if we look back in time, then Ω becomes ever closer to unity. The exact dependence during the matter dominated phase is given by Eq. (6.10):

$$\Omega = \frac{\Omega_0(1+z)}{1+\Omega_0 z}. \tag{13.1}$$

(In this equation, z is used instead of time t). The value of Ω is uncertain, but we know that the mass in observable stars and gas make $\Omega \geq 0.02$, and the dark matter imputed to be in galaxy clusters gives $\Omega \geq 0.2$. At the time of recombination, then,

Ω deviated from unity by at most 3% and possibly as little as 0.3%. At the time that "known physics" begins (of order 10^{-32} seconds), the value of Ω must be 1 to an accuracy of $10^{-43?}$. In standard cosmology there is no known way to make Ω be this close to unity; it must be input as an initial condition. Futhermore, for most of the history of the universe, Ω is either very close to 1 or very close to 0; if Ω is, in fact, of order 0.2 today, then we live at a very special time. This fact has led some astronomers to argue that since we probably don't live at a special time and Ω is certainly not close to 0, then it must be very close to 1.

B. Horizon Problem

In a universe with $\Omega > 0$, any observer can see only a limited portion of the universe at any given time. The size of this portion can be found by computing the proper distance that corresponds to infinite redshift. This is given by:

$$R_0 u = \int_{t_i}^{t_0} \frac{R_0 dt}{R(t)}. \tag{13.2}$$

For example, for $\Omega = 0$, we have

$$R_0 u = \int \frac{R_0 dt}{t} = (1/H_0) \ln\left(\frac{t_0}{t_i}\right) = \frac{\ln(1+z)}{H_0}. \tag{13.3}$$

For $\Omega = 1$, we find

$$R_0 u = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{13.4}$$

For $\Omega = 0$ we can barely see to infinite distance, while for $\Omega = 1$, we can see to a horizon limit of $2/H_0$. The total mass enclosed within the horizon is of order $1/H_0$. As we go back in time, H gets bigger and the amount of mass within our horizon gets smaller. This means that parts of the universe that are well within our horizon today were causally disconnected at very early epochs. In particular, the microwave background radiation coming from different portions of the sky was emitted at a time when those portions lay outside of each other's respective horizons; had they been at different temperatures initally, they could not have equilibrated. In short, the universe could not have evolved to the high degree of homogeneity and isotropy that we observe today; those properties too must have been imposed as an initial condition.

C. Inflation

The physics that lies behind the concept of inflation lies well outside the scope of this book; a good reference is "The Early Universe" by Kolb and Turner. Briefly, however, standard physics is inadequate to describe the nature of matter (in particular the equation of state) above temperatures of order 10^{27} K, and one must resort to so-called Grand Unification Theories, which attempt to unite the theories of the strong, weak, and electromagnetic forces. These theories conjecture that when the universe cools below a temperature of order $T = 10^{27}$ K, matter is trapped for a while in a state known as a false vacuum. The details need not concern us; what is relevant, however, is that the equation of state for such matter is given by $P = -\rho$ (rather than the conventional $P = \rho$ for ordinary relativistic matter). The major consequences of inflation follow directly from this equation.

The calculation of the behavior of the universe during the inflationary phase follows the same procedure as was used in chapter 8 for a universe of ordinary matter. The only extra step needed is that we need to determine the relation between ρ, and P as a function of radius R. This relation can be derived using the first law of thermodynamics. Take a sphere of matter with volume V $(= (4/3)\pi R^3)$. The total enclosed mass-energy is $E = \rho V$. The first law of thermodynamics is

$$dE = dQ - PdV, \tag{13.5}$$

where dQ is the heat flow into or out of the volume. For a homogeneous universe, $dQ = 0$ always (no temperature gradients). Then it immediately follows that

$$dE = \rho dV + V d\rho = -PdV. \tag{13.5}$$

Given an equation of state relating ρ and P, this equation can be integrated to give the relation between ρ and R. For the inflationary universe, this relation is quite simple. Inserting $\rho = -P$, we find that $d\rho = 0$, or $\rho = \rho_0 =$ constant. The value of ρ_0 comes from Grand Unification Theories.

The expansion equations are particularly simple to solve for the inflationary universe. Beginning with Eq. (8.2), we have

$$\ddot{R} = -\frac{4\pi G}{3}(\rho + 3P)R = \frac{8\pi G\rho_0}{3}R. \tag{13.7}$$

Integrating once gives

$$\frac{1}{2}\dot{R}^2 = \frac{4\pi G}{3}\rho_0(R^2 + R_C^2), \tag{13.8}$$

where the integration constant R_C is introduced in place of the previously used ϵ. The choice of a $+$ sign is appropriate for an open universe; a $-$ sign would be used for a closed universe. Only the open case will be worked out here, although the closed case can be solved in a comparable fashion. Note that R_C is *not* the initial radius at any particular time. If $R_C = 0$, we have a critically bound universe.

For convenience, let $\omega^2 = (8/3)\pi G\rho_0$.

One further integration gives:

$$\sinh^{-1}\left(\frac{R}{R_c}\right) = \sinh^{-1}\left(\frac{R_i}{R_c}\right) + \omega(t - t_i), \tag{13.8}$$

with integration constants defined so that $R = R_i$ at time $t = t_i$ that corresponds to the start of the inflationary epoch. For a closed universe, the sinh function is replaced by cosh.

The behavior of R with time is quite different from the previous radiation and matter dominated universes. At large time, Eq. (13.9) shows that R grows exponentially with time (hence the term "inflation").

Examination of Eq. (13.8) shows that once again the critical density for a marginally bound universe is given by

$$\rho_{crit} = \frac{3H^2}{8\pi G}. \tag{13.10}$$

We have

$$\Omega = \frac{\rho_0}{\rho_{crit}} = \frac{R^2}{R^2 + R_C^2}. \tag{13.11}$$

R_C/R_i is a measure of how close to critically bound the universe is at the beginning of the inflationary epoch. In fact, we have

$$\frac{R_C}{R_i} = \sqrt{\frac{1-\Omega_i}{\Omega_i}}. \tag{13.12}$$

If the ratio is small, the universe is close to critically bound; if large, the universe is open. However, Eq. (13.11) shows that as R increases and exceeds R_C, Ω approaches unity; furthermore, this process occurs exponentially fast. At the end of the inflationary epoch, when $R = R_f$, Ω has the approximate value

$$\Omega = 1 - \left(\frac{R_C}{R_f}\right)^2. \tag{13.13}$$

From Eq. (13.9) we see that

$$R_f \approx R_C \exp[\omega(t_f - t_c)] \tag{13.14}$$

regardless of the initial value of R_i provided $R_f >> R_C$ by the end of inflation.

The inflationary epoch lasts from 10^{-34} to 10^{-32} seconds, during which time the universe undergoes about 100 e-foldings, or a factor 10^{43} increase in size. Thus by the end, Ω differs from unity by a factor 10^{-86}. At 10^{-32} seconds, the universe undergoes a phase transition at which point the equation of state returns to that of ordinary radiation.

At any epoch, we have

$$\Omega - 1 \propto \frac{1}{R^2H^2}. \tag{13.15}$$

(This follows from the Lemaitre equation and the definition of Ω). At the end of the inflationary era, Ω starts increasing again at rate proportional to t (for the radiation dominated phase) or $t^{2/3}$ (for the matter dominated phase). The relevant times are: end of inflation, $t_f = 10^{-32}$; end of radation epoch, $t = 3\times10^{11}$; today, $t = 3\times10^{17}$. Combining, we find $\Omega_0 - 1 = 3 \times 10^{-39}$, so unless Ω_i was extremely small at the start of the inflationary phase, we expect the universe today to be extremely close to being critically bound. This solves the flatness problem.

The horizon problem is solved as follows. Equation (13.14) gives the growth of the universe during the exponential phase of the inflationary period. Just before the entry into the exponential growth phase, the universe is thermalized and made uniform out to a radius corresponding roughly to the horizon size $1/H$ at that time. We now demonstrate that this radius encloses all of the mass visible today and much more. At the start of the exponential growth phase, we have $H = \dot{R}/R = \omega$, so the comoving distance corresponding to the horizon at that time is

$$u_{max} = \frac{1}{HR_c} = \frac{1}{\omega R_c}. \tag{13.16}$$

This comoving distance corresponds to a linear distance today of

$$R_0 u_{max} = \frac{R_0}{\omega R_c.} \tag{13.17}$$

If R_r is the radius of the universe at the end of the radiation dominated era (at $t_r = 3 \times 10^{11}$), then we have $R_r/R_f = 5 \times 10^{21}$ and $R_0/R_r = 10^4$; hence $R_0/R_c = 5 \times 10^{68}$. Within factors of 2, we have

$$\frac{1}{\omega} = \frac{t_i}{t_0}\frac{1}{H_0}. \tag{13.18}$$

Combining and inserting in Eq. (13.17), we find that the universe is thermalized over a size today that is

$$\begin{aligned} R_{\text{Horizon}} =& R_0 u_{max} = \frac{R_0}{R_c}\frac{t_i}{t_0}\frac{1}{H_0} \\ \approx& 10^{17}\frac{1}{H_0}. \end{aligned} \tag{13.19}$$

This is enormously larger than the current horizon size, so we have solved the horizon problem.

Inflation also solves a problem unique to GUTs, namely the magnetic monopole problem (indeed, solving this problem was the original impetus for developing the process of inflation). Briefly, GUTs predicts the existence of massive magnetic monopoles which, if there were no inflationary epoch, would dominate the mass density of the universe today. Inflation reduces the density of monopoles to the point that their density today is negligible. Finally, inflationary theory predicts that at the end of the inflationary period, when the universe makes a state transition to that of ordinary matter, there will be a massive overproduction of matter relative to that of antimatter. The details are well beyond the scope of this book, and the reader is referred to a text such as *The Early Univerese* (Kolb and Turner) for more details.