IV. Robertson-Walker Metric

The goal of this section is to derive a metric for 4 dimensional space-time that describes an expanding universe and obeys the cosmological principle, the so-called Robertson-Walker metric.

A useful analogy for describing the expanding universe is to think of the surface of an expanding balloon that is covered with dots. The surface of the balloon represents a cosmological "fluid" (*i.e.*, the mass and radiation filling the universe). Each dot has a view of the universe that is identical to that of any other dot. The speed of recession of any dot relative to another obeys the Hubble law. Furthermore, each dot thinks that it is at rest relative to the motion of the nearby surrounding dots. Each dot can serve as a "fundamental observer" in that universe.

Not all observers in an ideal universe are equal. We define a "fundamental observer" (F.O.) to be an observer which is at rest with respect to the local cosmological fluid and hence has an isotropic view of the universe. It might seem peculiar that, given the high recession velocities of different parts of the universe relative to one another at the big bang, it still makes sense to talk about F.O.'s that are at rest in some sense. However, as we can show later, any peculiar motion that an observer has with respect to the F.O.'s dies out with time as long as the universe is smooth, and so in the absence of further accelerations, all observers become F.O.'s with time. By the Cosmological Principle, all F. O.'s see the same homogeneous, isotropic view of the universe. Even in an inhomogeneous universe, we can still establish the frame of an F. O. by observations of the microwave background.

Each F.O. carries its own clock that measures an absolute time since the Big Bang. In the space-time continuum, therefore, it makes sense to describe a space-like surface on which all F.O.'s measure the same absolute proper time t since the big bang. By the Cosmological Principle, all observers on this surface measure the same ρ, H_0, *etc.*at any time. Even if the F.O.'s have not been measuring absolute time, it is still possible to invent synchronization procedures by which they could establish a time whereby all measure the same physical parameters of the universe. This time is referred to as "cosmic time".

The above discussion suggests that a natural coordinate system for describing the universe is one in which the time coordinate is cosmic time t and the three space coordinates describe the space-like surface $t =$ constant. Our task is to find the most general form of the metric describing such a coordinate system that still obeys the cosmological principle.

For the space part, we wish to set up a spherically symmetric coordinate system centered on some particular F. O. The metric must be such, however, that if we rotate or translate the coordinate system, its form does not change. In 3-D Cartesian

space, for example, spherical coordinates (r, θ, ϕ) are valid, even though they seem to have a unique axis. If we pick another set of coordinates (r', θ', ϕ) centered on some other axis, however, the form of the metric is still the same.

This argument suggests that we should pick two of the coordinates to be the two angular coordinates (θ, ϕ) of a spherical coordinate system centered on some F.O. For convenience, define $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ as being the combination in which these coordinates will appear in the metric. These coordinates satisfy the condition of isotropy; that is if we choose new axes about which to measure θ and ϕ, then at least the form of $d\Omega^2$ does not change. To complete the coordinate system, it is necessary to specify a radial coordinate r. For now, this can be considered to be a function of the linear radius from the origin and of time.

With the above considerations, the metric must look like

$$-c^2 d\tau^2 = ds^2 = -A(r)c^2 dt^2 + B^2(r,t)dr^2 + C^2(r,t)d\Omega^2. \tag{4.1}$$

Following convention, we will take units such that $c = 1$ hereafter unless otherwise noted.

The form of the metric and the functions A, B, and C can be constrained as follows:

1. Isotropy demands that there be no cross-terms.

2. Although r has been left arbitrary up to now, we pick r to be a co-moving radial coordinate; *i.e.*, r labels a specific F.O. for all time. Hence the surface $r =$ constant labels a set of F.O.'s. If we want t to be our cosmic time, then we must have $A = 1$ (*i.e.*, the proper time measured along the path $r =$constant is given by $d\tau^2 = A(r)dt^2$, but since that is the path of an F.O. and the proper time measure by an F.O. is precisely the cosmic time t, we must have $d\tau = dt$ and so $A = 1$).

3. Everyone must measure the same Hubble constant at the same time t. For example, in the radial direction two galaxies are separated by $\Delta l = Br$. Two galaxies at fixed r but angular separation θ have a linear separation $\Delta l = C\theta$. Now $H = (1/\Delta l)(\partial \Delta l/\partial t)$. So $H = (1/B)(\partial B/\partial t) = (1/C)(\partial C/\partial t) = f(t)$ only. Hence B and C must have the form

$$B(r,t) = R^2(t) f_1(r), \tag{4.2a}$$
$$C(r,t) = R^2(t) f_2(r). \tag{4.2b}$$

Here, f_1 and f_2 are arbitrary functions.

4. Although r is a co-moving coordinate, it is otherwise an arbitrary function of the radial separation of two F.O.'s. Introduce a new radial coordinate λ by letting $d\lambda = f_1(r)dr$ and hence $\lambda = \int_0^r f_1(r')dr'$. Let $f_2(r) = a(\lambda)$. Note that λ is also a co-moving coordinate. It is specified to within an arbitrary zero-point and constant scale factor. For convenience, we take $\lambda = 0$ at the origin.

   Now the metric looks like
$$-d\tau^2 = ds^2 = -dt^2 + R^2(t)\left[d\lambda^2 + a^2(\lambda)d\Omega^2\right]. \tag{4.3}$$

5. The function $a(\lambda)$ cannot be arbitrary but is constrained by the requirement that if we change coordinates, then the form of the metric in the new coordinate system must be unchanged. Now we know that in two dimensions, the surface of a sphere has that property, and we wish to find the most general coordinate system that has that property. We have already introduced the concept of curvature as being a measure of the "spherishness" of a surface, and so let us see what happens by extending that concept. In particular, consider repeating the experiment of measuring the distance D between two nearly parallel geodesics. For our 3 dimensional space, it is enough to consider two geodesics that lie in the plane $\theta = \pi/2$ and that form an angle δ. The separation between the two lines is $D = Ra(\lambda)\delta$. The measure of curvature is $(1/D)(d^2D/ds^2) = (1/aR^2)(d^2a/d\lambda^2)$. On a sphere, this quantity has the value $-1/r_0^2$ where r_0 is the radius of the sphere, and in particular it is independent of the way in which the geodesics are picked out and of the origin of the coordinate system. More generally, then, we demand that this measure of curvature be constant everywhere; *i.e.*,
$$\frac{1}{aR^2}\frac{d^2a}{d\lambda^2} = -\frac{k}{\lambda_0^2}, \tag{4.4}$$
   where λ_0 is some scale length and $k = \pm 1$ or 0. The solution is

$$a = A\sin\lambda/\lambda_0 \quad k = +1 \tag{4.5a}$$
$$a = A\lambda \quad k = 0 \tag{4.5b}$$
$$a = A\sinh\lambda/\lambda_0 \quad k = -1. \tag{4.5c}$$

Remembering that λ still has an arbitrary scaling, this scale factor is now taken to be λ_0. A cannot be specified freely but must satisfy the condition that for λ small we require that the space be Cartesian. This requires that $A = \lambda_0$ for $k = \pm 1$ and $A = 1$ for $k = 0$. If we define yet another radial length coordinate $u = \lambda/\lambda_0$, the form for the metric is
$$ds^2 = -dt^2 + R^2(t)\lambda_0^2\left[du^2 + S_k^2(u)d\Omega^2\right], \tag{4.6}$$
where
$$S_k(u) = \begin{cases} \sin u & (k = +1) \\ u & (k = 0) \\ \sinh u & (k = -1). \end{cases} \tag{4.7}$$

Finally, the time-dependent radial scale factor $R(t)$ can be replaced by $R' = R\lambda_0$. Then, after dropping the prime, the metric has the final form

$$ds^2 = -dt^2 + R^2(t)\left[du^2 + S_k^2(u)d\Omega^2\right]. \tag{4.8}$$

The above discussion is not a rigorous proof that the coordinate systems so derived lead to an invariant metric; however, one can do so explicitly. More details are given in an appendix.

This is one form of the Robertson-Walker Metric. Another form can be found be letting $v = S_k(u)$, $dv = C_k(u)du$ (C_k being a cos or cosh as appropriate). Then

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dv^2}{1-kv^2} + v^2 d\Omega^2\right]. \tag{4.9}$$

These are called intrinsic or Schwarzschild coordinates.

The radial scale factor plays the role of "radius of curvature" for k non-zero. For $k = +1$, it is precisely the radius of the 3 dimensional surface of a 4 dimensional sphere. For $k = -1$, the space is hyperbolic, and so now it is sort of a "negative" radius of curvature. A hyperbolic surface cannot be embedded in any Cartesian space of one higher dimension. However, the two-dimensional version of a hyperbolic space can be approximated locally by a saddle-shaped geometry. For $k = 0$, there is no intrinsic scale length, and so R and u may be rescaled in tandem without changing the metric. For $k = \pm 1$, there is no arbitrariness in either u or R. (We will show later how R can be written explicity as a function of H_0 and q_0. If these numbers are measured and q_0 turns out to be different from 1/2, then the distance u to, say, the Virgo Cluster has a unique value.)

The geometries corresponding to $k = 1, 0$, and -1 are referred to as spherical, flat, and hyperbolic respectively. Flat space has conventional 3 dimesional Euclidean geometry. However, these terms can be misleading. It is important to remember that they refer to the geometry of a 3 dimensional surface that is embedded within a 4 dimensional space-time and that this 3 dimensional surface is defined by the requirement that it be a surface of constant proper time in an expanding universe. In the same way that the spherical geometry of a 2 dimensional sphere does not reflect the fact that it is embedded in a flat Euclidean 3 dimensional space, likewise the geometry of the 3 dimensional surface does not necessarily reflect the geometry of the 4 dimensional space-time as well. In particular, it turns out that in a Lorentzian space, the appropriate metric for an expanding "empty" universe is the one with hyperbolic geometry, even though we normally talk of a Lorentzian space as being one of flat space-time. The distinction is not particularly profound but can lead to confusion.

Geodesic Motion

In Chapter 2 we showed that if a particle follows a path such that the cumulative elapsed proper time is an extremum (actually, it is a maximum), then that path is one of constant velocity. Such a path is called a geodesic. Because proper time is an invariant (independent of the reference frame or coordinate system used to compute it), the methodology can be extended to compute the path of a freely falling particle in any coordinate system.

We wish to compute the path of a particle that is not a Fundamental Observer in an expanding universe . Let the particle have a "peculiar" velocity that is of order v, and let it travel a distance that is of order D in time $t \approx D/v$. The path can be broken up into N segments, each one passing in the vicinity of a Fundamental Observer. During the time that the particle is in the vicinity of the F. O., it undergoes a small acceleration $dv/dt \propto G\rho(D/N)$ that acts over a time $\Delta t \approx t/N$. The net change in velocity is $\Delta v \propto G\rho(Dt/N^2)$. For N large, this change is negligibly small, and the particle is on a path of constant velocity to very high accuracy. Thus, the elapsed proper time along the path in any one segment is again an extremum, and the same goes for the integrated proper time along the entire path.

Mathematically, the elapsed proper time is given by

$$\tau = \int \sqrt{1 - R^2\dot{u}^2}dt. \tag{4.10}$$

If we follow another path $u'(t) = u(t) + \delta(t)$, the change in proper time is given by

$$\delta\tau = \int \frac{R^2\dot{u}(d\delta/dt)}{\sqrt{1 - R^2\dot{u}^2}}dt. \tag{4.11}$$

Integrating by parts and making the demand that $\delta\tau = 0$ for for any arbitrary path deviation, we find that

$$\frac{d}{dt}\left[\frac{R^2\dot{u}}{\sqrt{1 - R^2\dot{u}^2}}\right] = 0. \tag{4.12}$$

The quantity in brackets is thus a constant. Since the local peculiar velocity of the particle is given by $v_{pec} = R\dot{u}$, we recognize the quantity in brackets as being RP, where P is the relativistic momentum per unit rest mass of the particle. For nonrelativistic velocities, this is just the normal linear velocity v. Thus, we see that the peculiar velocity decays as the universe expands in the proportion $v \propto 1/R$.

A Special Universe

The preceding concepts can be given a concrete visualization by considering the evolution of an empty universe. Take a Lorentz frame with spherical coordinates S,

θ, and ϕ and time coordinate T (For consistency with the Robertson-Walker metric, t will be reserved for another time). The metric has the form:

$$ds^2 = -dT^2 + dS^2 + S^2 d\Omega^2. \tag{4.13}$$

At $T = 0$ throw a bunch of particles away from the origin with a distribution of velocities. The particles can be considered to be massless galaxies. Look at one particle with velocity V. Along its world-line, it keeps a proper time t given by $t = T\sqrt{1 - V^2}$ or, squaring and using $VT = S$, $t^2 = T^2 - S^2$. Each particle is a Fundamental Observer in the cosmological sense, and so it makes sense to introduce a new time coordinate t; a surface $t =$ constant is formed by looking at all particles that have the same time t as measured by their own local clocks. Each particle can be labeled by its velocity V (and thus V could serve as a comoving coordinate), but a more useful coordinate u can be defined by letting $V = \tanh u$. Thus, we can use (u, t) as coordinates in place of (S, T); the transformation equations are $S = t \sinh u$, $T = t \cosh u$. What happens if we choose a different particle to represent $S = V = 0$? Then the cosmic time t remains unchanged and the space coordinate u is replaced by $u' = u - u_0$, where u_0 is the coordinate of the new origin. (This simple change of coordinates would not have been possible has we retained V as the space dimension.) The full 3-dimensional metric in the old and new coordinate systems is given by

$$ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2 = -dt^2 + t^2(du^2 + \sinh^2 u d\Omega^2). \tag{4.14}$$

Hence our judicious choice of u resulted in nothing less that the Robertson-Walker metric for $k = -1$ with $R(t) = t$.

There are several interesting points to note:

i) This hyperbolic space is just flat space-time written in a "funny" coordinate system.

ii) Proper distances in a surface t=constant cannot be measured by any single observer. A cooperative effort of many observers is required.

iii For very large t, a given linear distance coresponds to a small u: $r \approx ut$, $t \approx T$.

iv) The linear distance to any nearby particle is $r = ut$; its recession velocity is $\dot{r} = u$. The Hubble constant is defined to be $H = \dot{r}/r = 1/t$. For a general R-W metric, we have $H = \dot{R}/R$. Normally we use the subscript 0 to indicate the value of H at the present time t_0.

Here is thought to ponder. On a hyperbolic surface, if we draw a triangle, then the sum of the interior angles is less than 180 degrees. Indeed, that is the case if 3 F.O.'s make a triangle. Yet if we think of three stationary observers making the same triangle, they will measure the standard 180 degrees. What is going on?

To specify a complete cosmological model, it is necessary to determine the most general form of the function $R(t)$. Above, we showed that $R = t$ is the solution for a freely expanding universe. Next we shall look at what happens when the universe has a finite mass density.

At this point standard cosmological texts appeal to the full mechanics of General Relativity to find the equations for R vs. t. In the next chapter, these relations will be derived in a much simpler fashion by making direct use of the relations upon which GR is founded.