VIII. MODELS WITH FINITE PRESSURE

Up to now we have ignored radiation or other possible forms of matter that have substantial pressure. For example, at early times (*i.e.*, large z), radiation dominates the energy density of the universe. We wish to calculate both the time (or redshift) at which that occurs and the time evolution of the universe before that epoch. We also believe that today, a substantial portion of the matter in the universe is in a form known as "dark energy", which has a negative pressure.

It will be convenient to deal with these other forms of matter in one generalized treatment. The overall principles are the same, even though the details are different.

A. Equation of State

The equation of state for matter relates pressure to density (and internal temperature, if needed). In many cases of interest this relation can be parameterized quite simply:

$$P = w\rho c^2, \tag{8.1}$$

where w is a constant. (For clarity, the factor c^2 is included above to indicate explicitly how P and ρ are related dimensionally; it will be dropped hereafter.) For radiation, $w = 1/3$. For ordinary and cold dark matter, $w = 0$ to a good approximation. For a classic cosmological constant form of matter, $w = -1$. For simplicity we will only consider cases where w is constant, although some theories of dark energy do not require it to be so.

To derive the cosmological equations for this form of matter, it is necessary to determine the relation between ρ, and P as a function of radius R. This relation can be derived using the first law of thermodynamics. Take a sphere of matter with volume V [$= (4/3)\pi R^3$]. The total enclosed mass-energy is $E = \rho V$. The first law of thermodynamics is

$$dE = dQ - PdV, \tag{8.2}$$

where dQ is the heat flow into or out of the volume. For a homogeneous universe, $dQ = 0$ always (no temperature gradients). It immediately follows that

$$dE = \rho dV + V d\rho = -PdV. \tag{8.3}$$

Given the equation of state (8.1), this equation becomes

$$\rho(1 + w)dV = -V d\rho. \tag{8.4}$$

Integrating, we find

$$\rho = \frac{K}{V^{1+w}} = \frac{K}{R^{3(1+w)}}. \tag{8.5}$$

B. Equations of motion

Deriving the equation of motion for R is not as simple in a universe with substantial pressure as it was for a matter dominated universe. For matter dominated universe, the energy equation actually gives the energy per unit mass. Since mass is conserved, it is of no consequence if one works with energy per unit mass or total energy. In the present situation it is not obvious that energy per unit mass is conserved. For example, if the internal rest energy is changed due to, say, expansion, the kinetic energy would also change since the rest mass has changed. Nevertheless, it turns out that the Lemaitre equation Eq. (5.5) for matter dominated universes applies unchanged to pressure dominated universes. To demonstrate this fact, it is necessary to in a bit more detail about how energy is measured in relativistic dynamics.

Consider once again a sphere with radius R and internal energy density ρ_r. (For consistency with previous notation, we use ρ_r in place of u to denote energy density of radiation.) The Einstein equivalence principle of intertial and gravitational mass says that the gravitational potential on the surface of the sphere depends not at all on the properties of the stuff inside the sphere; hence the potential (relative to infinity) is $\Phi = -(4\pi/3)G\rho_r R^3$. Now add a shell to the sphere that has a thickness in comoving coordinates of Δu. The following considerations hold:

a) For $w > 0$, the total energy of the shell at infinity is 0, since ρV goes to 0 as R goes to ∞. For $w < 0$, the same is true as R go to 0. In either case, as the shell is brought to a finite radius, the increase in internal energy is balanced by work done on the shell (Eq. 8.2) so the sum of the two is always 0.

b) The following thought experiment shows that energy per unit rest mass is the right quantity to consider for the kinetic part of the equation. Consider a box of radiation with density ρ, volume V, moving with velocity v with respect to a laboratory frame. The total energy as measured in the laboratory frame is $E = \rho V(\frac{1}{2}v^2 + c^2)$. In the rest frame of the box, assume a portion of the rest energy $\Delta\rho c^2$ is lost due to isotropic blackbody radiation. In the laboratory frame, the moving box does not radiate isotropically; the leading edge appears to have a higher temperature than the mean, and the trailing edge has a lower temperature. To first order the energy radiated is the same as for the isotropic case, but to second order in v/c the radiation energy is slightly more. A convenient way to calculate the radiated energy is to make use of the fact that in relativistic dynamics, the quantities $E^2 - P^2c^2$ is an invariant. In the rest frame, the energy radiated is $E_0 = \Delta\rho V c^2$, and the momentum P_0 is 0. In the laboratory frame, the received momentum is $P_1 = \rho V v$. The received energy is $E_1 = \sqrt{(E_0^2 - P_0^2 + P_1^2)} \approx \rho V(c^2 + (1/2)v^2)$. Thus, the energy per unit rest mass ρV is the same both before and after.

c) The following thought experiment also show that energy per unit mass is the right quantity to consider for the gravitational part of the equation. Consider a box of radiation once again and now lower it (say, on a string) into a gravitational potential Φ. The box loses potential energy, which is collected at infinity by the entity lowering the box. The total energy retained at infinity is $E = -\rho V(c^2+\Phi)$ (i.e., it loses the rest mass energy but gains the released potential energy, since Φ is negative). Once again, let the box radiate an amount of energy $\Delta\rho V c^2$. Because of gravitational redshift [need to do the Einstein elevator experiment here!!!] the energy received at infinity is $\Delta E = \Delta\rho V(c^2 + \Phi)$. The net total energy (before plus after) is the rest energy $\rho V c^2$. The weight on the string is less, but the energy per unit mass is still balanced.

2. The total change in energy of the shell when it moved to the surface of the sphere is the sum of four components: kinetic energy, potential energy, internal mass-energy density, and PV work involved in compressing the shell.

Hence the total energy per unit mass of a shell is given by

$$E = \frac{1}{2}\dot{R}^2 + \Phi = \frac{1}{2}\dot{R}^2 - \frac{4}{3}G\pi\rho R^2 = \epsilon. \tag{8.6}$$

Finally, by taking the derivative of Eq. (8.6) and making use of Eq. (8.5), we arrive at the acceleration equation:

$$\frac{d^2r}{dt^2} = -\frac{4\pi}{3}G(1+3w)\rho r. \tag{8.7}$$

Since $w = P/\rho$, we finally have:

$$\frac{d^2r}{dt^2} = -\frac{4\pi}{3}G(\rho+3P)r. \tag{8.8}$$

An interesting paradox seems to arise if we try to track the total energy of the universe. Consider, for example, an expanding shell of pure radiation. As the shell expands, it performs work on the surrounding environs, losing internal energy as a consequence. However, all parts of the universe are equal, so the surrounding environs should also be losing energy, not gaining it. The paradox arises because energy per se is not an invariant from one inertial frame to the next. From the perspective of a volume at rest, the surrounding environs do gain energy. It is possible to introduce an artificial boundary to the universe that can accept or supply energy at will, and the total energy budget would be preserved. In an infinite universe, however, the concept of total energy is not well defined.

In subsequent chapters we will develop these equations further for different types of mass-energy at different phases of the universe.