Derivation of Schwarzschild Metric

It is possible to derive the Schwarzschild Metric, which describes the gravitational field around a point mass, in the weak-field limit using only special relativity and the equivalence principle, plus the imposition of spherical symmetry. The equivalence principle states, in effect, that inertial and gravitational accelerations are equivalent. A stationary observer in a gravitational field observes the universe in a manner that is equivalent to an observer being accelerated relative to an inertial frame. A carefully crafted thought experiment can link the two domains.

Consider an observer in an elevator car with height d that is accelerating at a rate g. Photons are emitted at the top of the car and received at the bottom. One can consider this situation as photons emitted from a rest frame (x, t) and received in a moving frame (x', t') that is moving with velocity

$$v = -g\Delta t,$$

where

$$\Delta t = d/c$$

is the time for the photon to transit the car. (The convention will be that this velocity is negative). Conceptually, in order to do calculations using special relativity, we consider the acceleration to commence immediately after the photon is released, and terminate just before the photon is received such that inertial frames can be established at both endpoints.

We wish to transfer coordinates from a stationary observer at the origin of the rest frame to an observer at the origin of the accelerating system using light signals; this helps circumvent difficulties in setting up global reference frames in the case of gravitating systems.

First, we transfer the time coordinate by broadcasting pulses from the origin of the rest frame and receiving them at the origin of the accelerating frame. For convenience, assume that the origins of the rest and accelerating frames coincide at $t = 0$. Let T be be the time interval between pulses in the rest frame. The first pulse arrives at $t = t' = 0$. The second pulse arrives at

$$t = T + x/c = T + (v/c)t.$$

In the accelerating frame, the arrival time is given by

$$t = (t' + vx'/c^2)/\sqrt{1 - (v/c)^2} = t'\gamma$$

(since $x' = 0$). We thus have

$$T = t'\sqrt{(1 - v/c)/(1 + v/c)}.$$

Now, t' is the metric time τ kept by a clock in the accelerating frame; thus, the time-like portion of the metric is given by

$$d\tau^2 = dT^2(1 + v/c)/(1 - v/c).$$

Next, we transfer the distance measured along the path of travel. We consider two pulses of light emitted at the same time in the rest frame, one at $x = 0$, the other at $x = X$. We consider reception in the accelerating frame along the line $t' = 0$. This line corresponds to

$$t = vx/c^2$$

in the rest frame. Reception of the second pulse occurs at

$$x = X + ct = X + vx/c.$$

In the accelerating frame, the point of reception is given by

$$x = (x' + vt'/c^2)/\sqrt{1 - (v/c)^2} = x'\gamma$$

(since $t' = 0$).

Now, x' is the metric distance s measuring by a standard yardstick in the accelerating frame; thus one portion of the space-like metric is given by

$$ds^2 = dX^2(1 + v/c)/(1 - v/c).$$

The X coordinate is of intermediate interest only; below it we will replace it with a different coordinate.

Transverse distances are unchanged: $y = y'$ and $z = z'$. However, we would like to introduce a polar coordinate system. Let two slightly nonparallel beams of light have a separation angle of θ in the rest frame. The equation of aberration is given by

$$\cos\theta' = \frac{\cos\theta - (v/c)}{1 - (v/c)\cos\theta}.$$

For small angles,

$$\theta' = \theta\sqrt{(1 + v/c)/(1 - v/c)}.$$

[The following may need more thinking] Select an origin such that the distance from the origin of the rest (and, at $t = 0$, the moving) frame is x. Let this be the radial coordinate of the polar system. A transverse distance perpendicular to the x axis is given by

$$\Delta y = x\Delta\theta = \Delta y' = x'\Delta\theta'.$$

My method for transferring distances from the rest to the moving frame gives

$$x' = X\sqrt{(1+v/c)/(1-v/c)}.$$

Using my relation between θ' and θ, I thus get.

$$x\Delta\theta = X\Delta\theta\frac{1+v/c}{1-v/c}.$$

I want to use θ, ϕ of the rest frame to be the angular coordinates in the moving frame, now imposing a spherical coordinate system, for which all the above calculations are valid for transformations along, say, the polar axis. I want my metric to have an angular part of the form

$$ds^2 = r^2 d\Omega^2.$$

I pick

$$r = X(1+v/c)/(1-v/c).$$

Thus, my x above becomes r. I replace X with r in the radial part of the metric. Finally, the metric has the form

$$ds^2 = -\Big(\frac{1+v/c}{1-v/c}\Big)dT^2 + \Big(\frac{1-v/c}{1+v/c}\Big)dr^2 + r^2 d\Omega^2.$$

I want $v < 0$ for my gravitational field situation. Thus,

$$v = [\Phi(r) - \Phi(\infty)]/c = -GM/rc.$$

and (to first order in v/c)

$$ds^2 = -(1 - 2GM/rc^2)dT^2 + \frac{dr^2}{1 - 2GM/rc^2} + r^2 d\Omega^2.$$

Note that this derivation is only good to 1st order in v/c, even though I have craftily written the final result in a form that actually holds good to all orders. More generally,

$$1 - 2GM/rc^2 = 1 + 2\Phi/c^2.$$