

Tracking bunches of particles

- Aside: units
- Periodic longitudinal boundary conditions
- Characterizing distributions
 - moments
 - correlation coefficients
 - histograms
- Statistical definition of emittance
- Generating matched beams
 - including longitudinally uniform beams
- Compensating for statistical artifacts

Aside: units

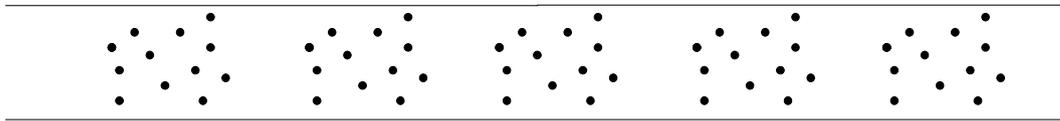
- Unit systems are the bane of accelerator physics
- Synergia uses two codes with two different systems of units
- $X_{impact} = U X_{chef}$
- $U_{xx} = U_{yy} = U_{tt} = \frac{\omega}{c}$
- $U_{x'x'} = U_{y'y'} = \gamma\beta$
- $U_{t't'} = -\gamma\beta^2$

units, cont.

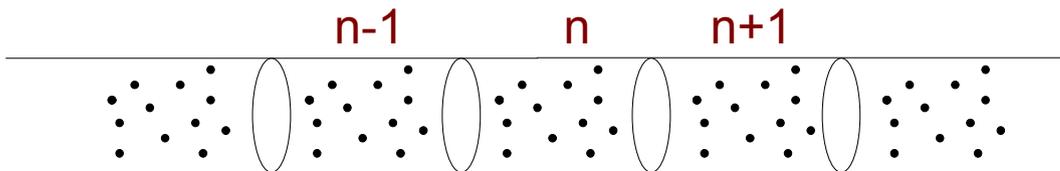
- The maps from gourmet are in impact units
- The chef unit of length is meters
- The impact unit of length makes more sense when you realize that when you track an RF system with frequency = scaling frequency, the longitudinal bucket is $\pm \pi$

Periodic longitudinal boundary conditions

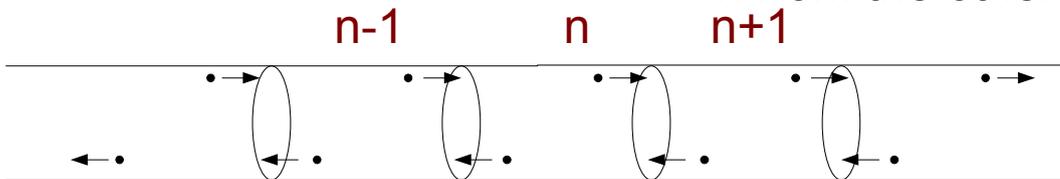
Take a long periodic (or uniform) beam



Look at period n



Each particle leaving n corresponds to another particle entering n from the other side



Characterizing distributions

- If we are going to deal with large numbers of particles, we need to have a set of observables by which to characterize the distribution

- First moment (mean) $\langle x_i \rangle$

- Second moment C:

- $$C_{ij} \equiv \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle$$

- Correlation coefficient R:

$$R_{ij} \equiv \frac{C_{ij}}{\sqrt{C_{ii}C_{jj}}}$$

characterizing, cont.

- These quantities are both useful in simulations and closely related to typical measuring devices
 - Beam position monitors report $\langle x_i \rangle$
 - Beam sizes are usually characterized by a width $\sim \sqrt{C_{ii}}$
- One-dimensional histograms are also useful and similar to the output of beam profile monitors.
- Of course, in the simulation we both have access to *and require* information not readily available in experiments.

Statistical definition of emittance

- $\epsilon = \sqrt{\det C_{ij}}$
 - note that this can be 2D (horizontal, vertical or longitudinal), 4D (transverse) or 6D (full phase space)
- Confusion danger: different conventions abound
 - This is “one sigma” ~ 36%
 - Unnormalized
- Numerical danger: taking determinants of even modestly-sized matrices is numerically tricky.

Generating matched beams

- Good news: real beams are usually well modeled by a Gaussian distribution in each coordinate
 - Gaussians are easy to generate. See, e.g., NR.
- Bad news: real beams contain correlations
 - Simplest case: 1D Courant-Snyder
 - $R_{xx'} = \frac{\alpha}{\sqrt{1 + \alpha^2}}$
 - Beams with couplings between planes are much more complicated.

Aside: random number generators

- Computing (pseudo-) random numbers is difficult
- Canned routines can be bad
 - e.g., Intel Fortran ca 2002 (I haven't checked it lately)
- Ideally, you should always know which random algorithm you are using.
- You must always know how your random seed is being set.

Matching

- A matched beam has the same distribution after each period (turn, etc.)

$$x_f = M x_i$$

$$\sum x_f x_f^T = \sum M x_i x_i^T M^T$$

$$C \equiv \sum x_f x_f^T = \sum x_i x_i^T$$

$$C = M C M^T$$

$$M e = \lambda e \Rightarrow \{\lambda_i, e_i\}$$

$$E_i \equiv e_i e_i^\dagger$$

Given a set of constraints (widths, emittances, etc.), the problem is then to solve for the a_i

$$C = \sum_i a_i E_i$$

Subtleties

- Will the given procedure always work?
 - Only if the all degrees of freedom, including longitudinal, are “matchable”
 - i.e., stable RF
 - If the procedure fails, the components E_i will fail to span the space:
$$\det \left(\sum_i E_i \right) = 0$$
 -
- What if we have a uniform longitudinal beam or a bunching/debunching beam?
 - One possibility: only match transversely
 - Neglects to compensate for dispersion

Special case: uniform longitudinal beam

- If we are modeling a uniform longitudinal beam with periodic boundary conditions, the map does not tell the whole story.
 - And if we are not using periodic boundary conditions, the beam will not stay uniform!
 - If the longitudinal coordinate is uniform, symmetry prevents any couplings to other coordinates. All couplings to the spatial longitudinal coordinate (other than the self-coupling) should be set to zero in the map for the purposes of matching.

Fixing the distribution

- It is easy to generate a set of uncorrelated vectors $\{v\}$ with Gaussian distribution with unit widths:

$$\langle v_i v_j \rangle = \delta_{ij}$$

- We want a set of vectors $\{r\}$ with correlations given by the C from our procedure.
- Solution: $r_j = G_{jk} v_k$
- where $C = GG^T$
- GG^T is the Cholesky decomposition of C

Correcting for finite statistics

- In reality, our set of random vectors will have statistical fluctuations. These fluctuations are non-physical because we are typically using n macroparticles $\ll N$ physical particles
- If we have $\langle v \rangle = \bar{v}$
- and $\langle v_i v_j \rangle = X_{ij}$
- then another Cholesky decomposition $X = HH^T$
- allows the solution $r_j = A_{jk}(v_k - \bar{v})$
- with $A = GH^{-1}$