HW 4; Due Thursday, March 7

1. Assume that there is only matter and radiation in the universe (no cosmological constant) and that the universe is flat ($\rho_0 = \rho_{\rm cr}$). Find the age of the universe when the cosmic temperature was 1 MeV and 1/3 eV.

2. Define the time-dependent Ω as

$$\Omega(t) \equiv \frac{8\pi G\rho(t)}{3H^2(t)}$$

where ρ counts the energy density in matter and radiation (assume zero cosmological constant). Suppose Ω is equal to 0.3 today. Find $\Omega(t) - 1$ as a function of the scale factor. How close to one would $\Omega(t)$ have been back at the Planck epoch (assuming no inflation took place so that the scale factor at the Planck epoch was of order 10^{-32})? This fine-tuning of the initial conditions is the flatness problem. If not for the fine tuning, an open universe would be *obviously* open (i.e. Ω would be almost exactly zero) today.

3. Compute the equilibrium number density of a species with mass m and degeneracy $g = 2$ in the limits of large and small m/T. Use Boltzmann statistics; i.e. take $f = e^{-E(p)/T}$, with $E(p) = \sqrt{m^2 + p^2}$. Compare these limits with the corresponding limits for Bose-Einstein and Fermi-Dirac statistics. In all cases, you can simplify enormously by taking the appropriate limit *inside* the integral; e.g. set $E(p) \to m + p^2/2m$ in the low T limit. Also you should find the relations

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x - 1} = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x + 1}$$

useful, where ζ is the Riemann zeta function and Γ the gamma function.

4. Find and apply the metric, Christoffel symbols, and Ricci scalar for a particle trapped on the surface of a sphere with radius r.

(a) Using coordinates t, θ, ϕ, the metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

Note that, in this case, the indices μ and ν range from 0 to 2 since there are only two spatial dimensions. Show that the only non-vanishing Christoffel symbols are ${\Gamma^\theta}_{\phi\phi}, {\Gamma^\phi}_{\phi\theta}$, and ${\Gamma^\phi}_{\theta\phi}$. Express these in terms of θ.

(b) Use these and the geodesic equation to find the equations of motion for the particle.

(c) Find the Ricci tensor. Show that contraction of this tensor leads to

$$\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu} = \frac{2}{r^2}.$$