TASI Lectures: Cosmology I

Definitions

1. Expansion rate, $H \equiv (da/dt)/a$.
2. Critical Density, $\rho_{\rm cr} \equiv 3H_0^2/(8\pi G)$
3. Fractional Energy Density of species i: $\Omega_i \equiv \rho_i/\rho_{\rm cr}$

Numbers

1. $H_0 \equiv 100h$ km sec^{-1} Mpc^{-1}; $h = 0.73 \pm 0.03$
2. $\rho_{\rm cr} = 1.879h^2 \times 10^{-29}$ g cm$^{-3} = 8.098h^2 \times 10^{-11}$eV^4
3. $\Omega_b h^2 = 0.0223^{+0.0007}_{-0.0009}$
4. $\Omega_m h^2 = 0.127^{+0.007}_{-0.013}$ with Ω_m summing contributions from all non-relativistic matter including dark matter and baryons.
5. Present temperature of cosmic microwave background (CMB), $T_0 = 2.725$ K $= 2.348 \times 10^{-4}$ eV
6. $\Omega_{\rm CMB} = 2.47 \times 10^{-5} h^{-2}$
7. Fractional energy density of three massless neutrinos, $\Omega_\nu = 1.68 \times 10^{-5} h^{-2}$

Equations

1. Geodesic Equation
$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$
where λ is parameter which monotonically increases along the space-time path of the particle.
2. Christoffel Symbol
$$\Gamma^\mu{}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right].$$
3. Friedmann-Robertson-Walker metric in a flat universe
$$g_{\mu\nu} = \mathrm{diag}\left(-1, a^2, a^2, a^2\right)$$
4. Friedmann equation in a flat universe
$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

Exercises

1. In 2D Euclidean space, start from the geodesic equation in cartesian coordinates

$$\frac{d^2x^i}{dt^2} = 0$$

where $x^1 = x$ and $x^2 = y$ and derive the geodesic equation in polar coordinates, $\tilde{x}^1 = r, \tilde{x}^2 = \theta$. Begin by inserting $dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$ into the cartesian equation.

2. Find the Christoffel symbols in 2D polar coordinates in Euclidean space. Start from the metric: $g_{ij} = \text{diag}(1, r^2)$. The only ones which are non-zero are: ${\Gamma^2}_{12}, {\Gamma^2}_{21}$, and ${\Gamma^1}_{22}$. Use these to find the two components of the geodesic equation. Show that this yields the correct equations for the evolution of r and θ.

3. Derive the time component of the geodesic equation for a massless particle traveling in an FRW metric. Because the particle is massless, its momentum four-vector

$$P^\mu \equiv \frac{dx^\mu}{d\lambda} = (E, P^i)$$

satisfies

$$g_{\mu\nu} P^\mu P^\nu = 0.$$

Use this to express $\delta_{ij} P^i P^j$ (which will emerge on the right-hand side of the geodesic equation) in terms of energy E. To eliminate the vague parameter λ, write

$$\frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{d}{dt} = \frac{dx^0}{d\lambda}\frac{d}{dt} = E\frac{d}{dt}.$$

4. Derive the Freidmann equation from Einstein's Eqn:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta} = 8\pi G T_{\mu\nu}.$$

Here the Ricci tensor $R_{\mu\nu}$ can be expressed in terms of the Christoffel symbols:

$$R_{\mu\nu} = {\Gamma^\alpha}_{\mu\nu,\alpha} - {\Gamma^\alpha}_{\mu\alpha,\nu} + {\Gamma^\alpha}_{\beta\alpha}{\Gamma^\beta}_{\mu\nu} - {\Gamma^\alpha}_{\beta\nu}{\Gamma^\beta}_{\mu\alpha}.$$

The Friedmann equation is the time-time ($\mu = \nu = 0$) component of the Einstein equation applied to the FRW metric. The time-time component of the stress-energy tensor T_{00} is simply equal to the energy density.