TASI: Cosmology IV

Definitions

1. Temperature anisotropy, Θ, varies with incoming direction, $\hat{n}$. This can be expanded in terms of spherical harmonics

$$\Theta(\hat{n}) = \sum_{lm} \Theta_{lm} Y_{lm}(\hat{n}).$$

2. Using orthogonality of the spherical harmonics, the moments can be expressed as

$$\Theta_{lm} = \int d^2n Y^*_{lm}(\hat{n})\Theta(\hat{n}).$$

The monopole is Y_{00} and the three components of the dipole Y_{1m}, etc. Before recombination, the monopole and dipole are significantly larger than the higher moments. This is a characteristic feature of a fluid.

3. Theories do not make predictions for particular moments Θ_{lm}. Rather they predict the distribution from which these moments are drawn; in particular they predict the variance

$$\langle \Theta_{lm} \Theta^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l.$$

Think of the C_l's as the variance in the CMB fluctuations on an angular scale of order l^{-1}. Note that the variance is identical for all m's corresponding to a given l. Therefore, for each l we can measure $2l + 1$ numbers to try to estimate the variance from which the numbers are drawn. So for low l it is hard to accurately estimate C_l. Quantitatively, there is a *cosmic variance* of $\sqrt{2/(2l + 1)C_l}$ associated with each C_l. no matter how accurately you measure the Θ_{lm}'s you can't extract a more accurate measurement of C_l than the floor set by cosmic variance.

Exercises

1. The speed of sound of the coupled electron-proton-photon fluid is

$$c_s^2 \equiv \frac{1}{3(1 + 3\rho_b/4\rho_\gamma)}.$$

Plot the sound speed as a function of redshift before decoupling at $z = 1089$.

2. The comoving sound horizon is the total comoving distance travelled by a sound wave. It sets the fundamental scale physical size of hot and cold spots in the CMB. Using the result from the first problem, compute the sound horizon at recombination:

$$r_s(\eta_*) \equiv \int_0^{\eta_*} d\eta' c_s(\eta').$$

3. Plot the sound horizon at recombination as a function of baryon density.
4. Compute the power spectrum of anisotropies today from the inhomogeneities on the last scattering surface.

   (a) Assume that the photons we see today from direction $\hat{n}$ come from the surface of last scattering: $\Theta(\vec{x}_0, \hat{n}, \eta_0) = (\Theta_0 + \Psi)(\vec{x} = \chi_* \hat{n}, \eta_*)$ where x_0 is our position, χ_* is the comoving distance to the last scattering surface, and η_* is the conformal time at last scattering. Fourier transform the right-hand side and expand the left in terms of spherical harmonics to get

$$\sum_{lm} \Theta_{lm} Y_{lm}(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\hat{n}\chi_*} (\tilde{\Theta} + \tilde{\Psi})(\vec{k}, \eta_*).$$

   Now expand the exponential using

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kx) P_l(\hat{k}\cdot\hat{x})$$

   where j_l is the spherical Bessel function and P_l the Legendre polynomial. Expand P_l as a sum over products of spherical harmonics. Then, equate the coefficients of $Y_{lm}(\hat{n})$ to get an expression for Θ_{lm}.

   (b) Square the Θ_{lm} you got in **(a)** and take the expectation value to get an expression for C_l. You should find

$$C_l = \frac{2}{\pi} \int_0^{\infty} dk\, k^2 P(k) j_l^2(k\chi_*) \left(\frac{\tilde{\Theta}_0(k, \eta_*) + \tilde{\Psi}(k, \eta_*)}{\tilde{\delta}(k, \eta_0)} \right)^2.$$

   Here $P(k)$ is the power spectrum of the matter overdensity, δ, today.
5. On large scales, the ratio in brackets in the previous problem is

$$\frac{\tilde{\Theta}_0(k, \eta_*) + \tilde{\Psi}(k, \eta_*)}{\tilde{\delta}(k, \eta_0)} = -\frac{c}{k^2}$$

   where c is a constant. [You can think of this as following from Poisson's equation which relates potentials (in the numerator) to overdensities (in the denominator).] Compute the large angle anisotropy spectrum, the so-called Sachs-Wolfe effect. Equation 6.574.2 from Gradshteyn and Rzyhik will be useful. Take the power spectrum to scale as k^n, and set $n = 1$ which corresponds to a scale invariant spectrum. An analytic form also exists for $n \neq 1$.