Dodelson: Exercise 7.11

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This note details a solution to Exercise 7.11 in Dodelson's "Modern cosmology" book. I show that Dodelson's equation (7.77) for the growth factor is only correct if the energy budget is entirely composed of matter, cosmological constant or curvature. I then numerically integrate the growth factor evolution equation for dark energy models with $w \neq -1$ and show that there is a $\sim 10\%$ disagreement with (7.77). Finally, I comment on some of the features of growth in the presence of dark energy with $w = 0$.

I. GROWTH FUNCTION

The starting point for the derivation of the growth function are the equations ($\dot{} = d/d\eta$)

$$\begin{aligned} \dot{\delta} + ikv &= -3\dot{\phi}, & (1)\\ \dot{v} + \frac{\dot{a}}{a}v &= ik\phi, & (2)\\ k^2\phi &= 4\pi G a^2 \rho_{dm}\delta, & (3) \end{aligned}$$

where we have neglected radiation, assumed that $aH \ll k$ so that modes are well inside the horizon, and assumed that any sources of energy other than matter and radiation do not cluster ($\rho_X \delta_X \ll \rho_{dm}\delta$). From equations (1) we can derive the second order evolution equation for δ ($' = d/da$)

$$\delta'' + \left(\frac{3}{a} + \frac{d\log H}{da}\right)\delta' - \frac{3\Omega_m H_0^2}{2a^5H^2}\delta = 0, \tag{4}$$

where we have neglected terms of order $O\left([aH/k]^2\right)$. Note that sources of energy other than matter enter into (5) only through the Hubble parameter (i.e. through the density). We wish to solve this to find the linear growth factor G(a), defined by $\delta(k,a) \equiv \delta_0(k)G(a)$ and given by

$$G'' + \left(\frac{3}{a} + \frac{d\log H}{da}\right)G' - \frac{3\Omega_m H_0^2}{2a^5H^2}G = 0. \tag{5}$$

II. VALIDITY OF DODELSON'S SOLUTION

First, lets analyse the validity of Dodelson's equation (7.77). Direct substitution into (5) of Dodelson's solution

$$G(a) = \frac{5}{2}\Omega_m \frac{H(a)}{H_0}\int_0^a \frac{da'}{(a'H(a')/H_0)^3}, \tag{6}$$

shows that (6) is a solution of (5) if $G(a) = 0$ (useless for structure formation) or if

$$\left[H'' + \left(\frac{3}{a} + \frac{d\log H}{da}\right)H' - \frac{3\Omega_m H_0^2}{2a^5H^2}H\right] = 0, \tag{7}$$

i.e. if H is a solution of (5). To check if this is true, we rewrite (7) in terms of $E(a) \equiv H^2/H_0^2$ giving

$$E'' + \frac{3}{a}E' - 3\frac{\Omega_m}{a^5} = 0. \tag{8}$$

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Consider functions of the form $E(a) = \Omega_m a^{-3} + f(a)$ i.e. matter plus some other component. This leads to the requirement

$$f'' + \frac{3}{a} f' = 0. \tag{9}$$

Considering a solution of the form $f \propto a^n$ gives the polynomial $n(n+2) = 0$, which has solutions $n = 0$ (cosmological constant) and $n = -2$ (curvature). Hence H is only a solution to (5) if, in addition to matter, we have a cosmological constant, curvature or a mixture of the two. This will not be true for a general dark energy component.

III. GROWTH FUNCTION - GENERAL CASE

Having disillusioned ourselves of the general validity of Dodelson's solution, we'll integrate (5) numerically to calculate the growth function and see how well (6) works in the case of dark energy with $w \neq -1$.

The set of equations to be integrated is

$$\begin{aligned}
u'(a) &= -\left(\frac{3}{a} + \frac{1}{2}\frac{d\log E(a)}{da}\right) u(a) + \frac{3\Omega_m}{2a^5 E(a)} G(a), && (10)\\
G'(a) &= u(a) && (11)\\
E(a) &\equiv \frac{H^2}{H_0^2} = \frac{\Omega_m}{a^3} + \frac{1-\Omega_m-\Omega_X}{a^2} + \frac{\Omega_X}{a^{3(1+w)}}. && (12)
\end{aligned}$$

With initial conditions set by the requirement that $G(a) = a$ in the matter dominated regime, i.e.

$$\begin{aligned}
u(a_0) &= 1, && (13)\\
G(a_0) &= a_0, && (14)
\end{aligned}$$

at some a_0 well into the matter dominated regime (I take $a_0 = 10^{-4}$). At this stage, I've assumed the dark energy equation of state w to be constant.

Figure 1 compares the result of the numerical integration $G_N(a)$ and (6) $G_{Dodelson}$(a) in the case of a cosmological constant ($w = -1$) and dark energy with $w = -0.5$. At early times $a \ll 1$ the two growth factors agree, but by the present day $G_N(a)$ and (6) $G_{Dodelson}$ differ by nearly 10%. Figure 2 amplifies on this by plotting $G(a = 1)$ as a function of w. To give a handle on the significance of this error, Figure 3 compares several different cosmologies indicating the scatter in the growth factor with cosmological parameters.

IV. GROWTH FUNCTION - $w = 0$

The case of $w = 0$ is a little curious in the above plots. Naively, one might think that in this limit the EdS universe would be recovered. This is not the case because the dark energy component is assumed to be perfectly homogenous i.e. there is no dark energy perturbation term in (3). If we assume flatness, $\Omega_m + \Omega_X = 1$ and $w = 0$ we can solve for the growth factor analytically. In this case, $H = H_0 a^{-3/2}$ and (5) reduces to

$$G'' + \frac{3}{2a} G' - \frac{3\Omega_m}{2a^2} G = 0. \tag{15}$$

Substituting $a = e^u$ gives a second order ODE with constant coefficients. Solving and rewriting in terms of a gives

$$G(a) = A a^{(-1/4+\sqrt{1+24\Omega_m}/4)} + B a^{(-1/4-\sqrt{1+24\Omega_m}/4)}. \tag{16}$$

Discarding the decaying solution and setting $A = 1$ we obtain

$$G(a) = a^{-1/4+\sqrt{1+24\Omega_m}/4}. \tag{17}$$

In the case that $\Omega_m = 1$ we recover $G(a) = a$, but note that as $\Omega_m \to 0$, $G(a) \to$ constant and growth is suppressed. Setting initial conditions in this $w = 0$ case is somewhat complicated (and not addressed in the above calculations) as the standard matter dominated growth function is never realised.

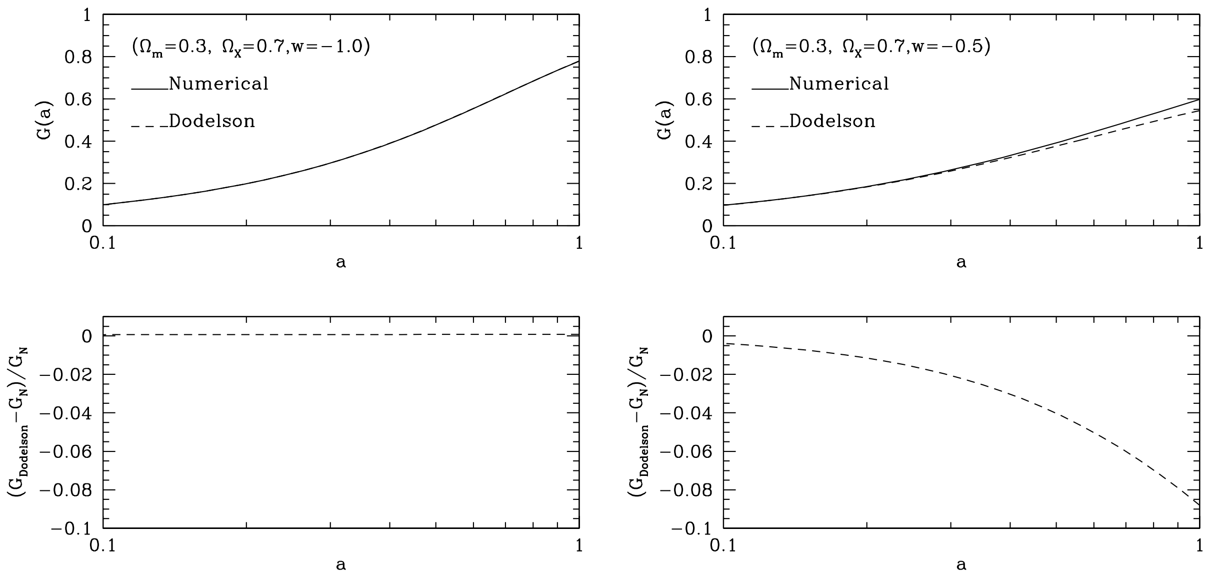

FIG. 1: Comparison of (6) and numerical calculation of the growth factor as a function of scale factor a ($\Omega_m = 0.3, \Omega_X = 0.7$). *Left panel:* $w = -1$.*Right panel:* $w = -0.5$. Numerical errors are at the 0.05% level. The right panel indicates that by the present day using (6) leads to a nearly 10% error in calculating $G(a)$.

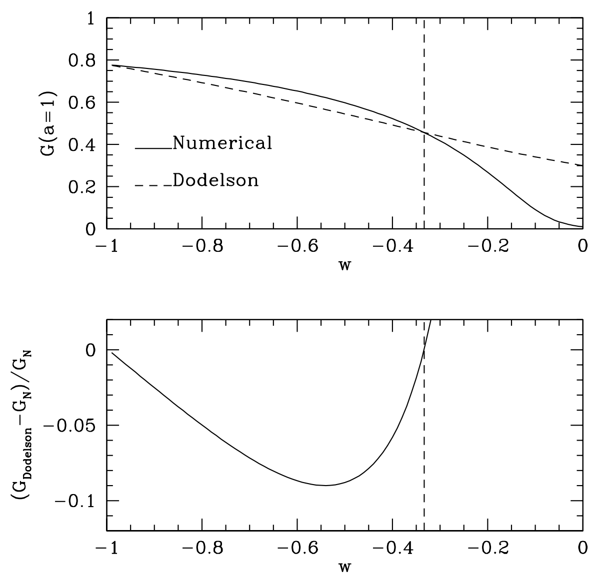

FIG. 2: Comparison of (6) and numerical calculation of the growth factor, evaluated at the present day ($a = 1$), as a function of w ($\Omega_m = 0.3, \Omega_X = 0.7$). We expect the two should agree at $w = -1$ (cosmological constant) and $w = -1/3$ (curvature, indicated by vertical line). They should not agree at $w = 0$, which is not equivalent to an Einstein-de Sitter universe, because the dark energy is assumed to be uniformly distributed, i.e. there is no dark energy perturbation term in (3).

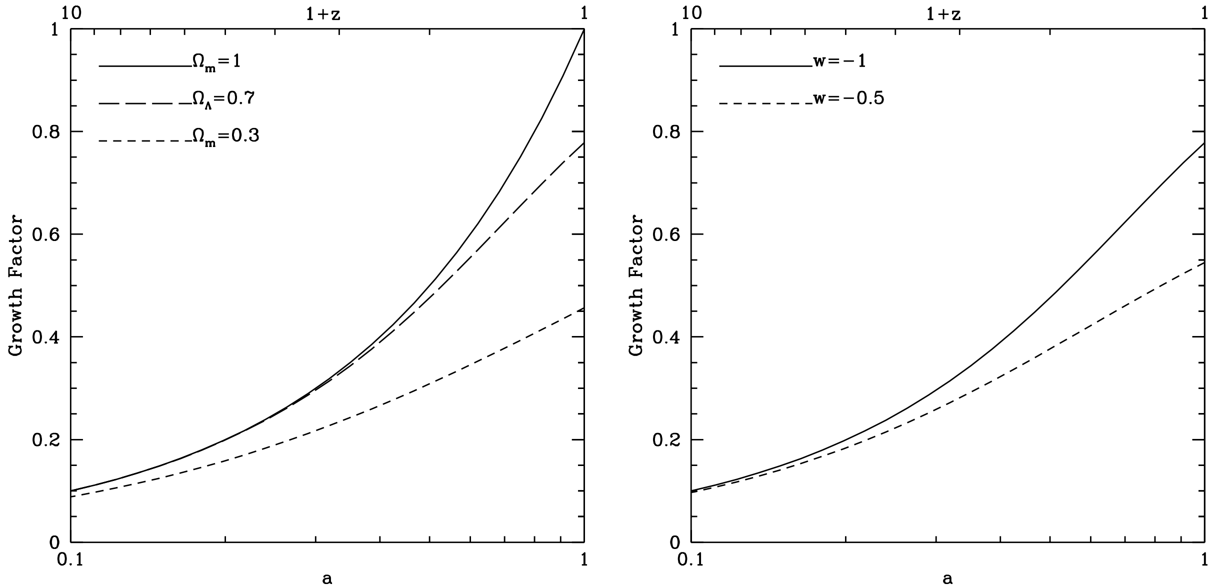

FIG. 3: *Left panel:* Growth factor as a function of a for three simple cosmologies c.f. Dodelson Figure 7.12. The first two are flat, while the $\Omega_m = 0.3$ universe is open. *Right panel:* Growth factor for $w = -1$ and $w = -0.5$ cosmologies ($\Omega_m = 0.3, \Omega_X = 0.7$). Calculated using numerical code. Note that the difference between the two dark energy cosmologies is significantly greater than the error from using the incorrect (6).