The Acceleration of Particles Using Time-Varying EM Fields

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I. INTRODUCTION

In the early part of last century, scientists realized the usefulness of particle accelerators. The first man-made accelerators employed electrostatic fields; in 1926 William Coolidge accelerated a beam of electrons to 900 keV using a series of high voltage X-ray tubes [1]. The maximum particle energy in electrostatic accelerators is directly proportional to the maximum voltage of these machines. Maximum voltages are on the order of megavolts yielding electron energies around an MeV, quite far from the high energies of today's electron accelerators $\sim$ 100 GeV.

Modern linear accelerators (*linacs*) use time-varying EM fields to continuously transfer energy from the fields to the particles. To accelerate the particles, one must have a component of electric field in the intended direction of acceleration and a field phase velocity v_p equal to the velocity of the accelerated particles $c\beta$. The first requirement is satisfied by the use of TM fields in a waveguide. The second requirement $v_p = c\beta$ is not satisfied in a uniform waveguide, but can be met by introducing periodic structure into the waveguide[2].

II. ENERGY GAIN IN A TIME-VARYING FIELD

One can find the energy imparted to an electron moving along the axis $r = 0$ with velocity $\mathbf{v} = v(z)\hat{\mathbf{z}}$ through a simple standing wave with an electric field [2]

$$\mathbf{E}(r = 0, z, t) \;=\; E(z)\; cos(\omega t + \phi)\; \hat{\mathbf{z}}. \tag{1}$$

Setting $t = 0$ when the electron enters the accelerating gap of length L and assuming that the phase of the wave $|\phi| \leq \pi/2$, i.e. the electron crosses the gap during the accelerating (positive) phase of the wave. (We will see later that we need $-\pi/2 \leq \phi \leq 0$ to maintain synchronization.) The electron feels the field above but with $t = t(z) = \int dz/v(z)$. The

energy gain is

$$\Delta U \;=\; e\int_{-L/2}^{L/2} E(z)cos(\omega t(z)+\phi)dz. \tag{2}$$

We can can define the average E-field amplitude

$$E_0 \;\equiv\; \frac{1}{L}\int_{-L/2}^{L/2} E(z)dz \tag{3}$$

and *transit-time factor*

$$T \;\equiv\; \frac{\int_{-L/2}^{L/2} E(z)\;cos(\omega t(z))\;dz}{E_0 L} - tan(\phi)\;\frac{\int_{-L/2}^{L/2} E(z)\;sin(\omega t(z))\;dz}{E_0 L}. \tag{4}$$

Using the indentity $\cos(\omega t+\phi) = \cos\omega t\;\cos\phi - \sin\omega t\;\sin\phi$, we can succinctly contain the physics in the *Panofsky equation*:

$$\Delta U \;=\; eE_0 T\cos(\phi)L \tag{5}$$

This is just the energy gain for an electron in a electrostatic field of voltage E_0 over a distance L, reduced by the factor $\cos(\phi)T$. Even for an ideal phase $\cos(\phi = 0) = 1$, one can see that $T \leq 1$, and that the energy gain in an RF field is less than that for a simple static field. There are many models for T each suggesting a specific tuning of the wave frequency ω and gap length L for maximizing ΔU, but the details are not specific to classical electromagnetic theory.

This reduction in energy gain is greatly outweighed by the advantages of a time-varying field: Assuming proper choice of gap lengths and wave frequency, these RF gaps can be linked in series to accelerate particles nearly indefinitely.

III. STANDING WAVES IN RESONANT CAVITIES

With only small beam holes bored on axis in the endplates, it is still instructive to view the waves inside a resonant accelerating cavity as standing waves. The simple *Alvarez structure* uses the lowest TM standing wave mode to accelerate particles through the cavity. A series of conducting drift tubes are installed inside the cavity to shield the particles from the negative phase of the standing wave. The drift tubes monotically increase in length to account for the increasing velocity of the particles [3].

For a cylindrical cavity of radius $r = R$, we can determine the fields for TM waves from a reworked form of Maxwell's equations

$$\mathbf{E}_t = \pm\frac{ik}{\gamma^2}\nabla_t\psi \tag{6}$$

$$\mathbf{H}_t = \pm\frac{\epsilon_0\omega}{k}\hat{\mathbf{z}}\times\mathbf{E}_t \tag{7}$$

$$E_z = \psi e^{\pm ikz} \tag{8}$$

with ψ as the scalar solution to the Helmholtz equation for $\gamma^2 = \mu\epsilon\omega^2 - k^2$ and $\nabla_t^2 = \nabla^2 - \partial/\partial z$:

$$(\nabla_t^2 + \gamma^2)\psi = 0 \tag{9}$$

$$\psi = J_m(\gamma_{mn}\rho)e^{\pm im\phi}. \tag{10}$$

For TM waves, boundary conditions require that $\psi(\rho = R) = 0$, and so $\gamma_{mn} = x_{mn}/R$ where x_{mn} are the n roots of the mth Bessel function. Boundary conditions also demand that E_t vanish at the endplates ($z = 0, L$), so the z-dependence $e^{\pm ikz}$ becomes $\cos(p\pi z/L)$ with $p = \{0, 1, 2...\}$ [4]. Notice the use of $\cos(p\pi z/L)$ instead of $\sin(p\pi z/L)$ even though the derivative in Eq. (6) is independent of z. This is because this form of Maxwell's equations (Eqs. (6,7,8)) were derived assuming an $e^{\pm ikz}$ z-dependence. One must look earlier in the derivation to see how the z-dependence changes between longitudinal and tranverse components; specifically, the choice becomes clear upon considering equations 8.23 in Jackson[4].

Using Eqs. (8 & 10) and the reconsidered z-dependence, we find that

$$E_z = E_0 J_m(\gamma_{mnp}\rho)e^{\pm im\phi}\cos(\frac{p\pi z}{L})\ e^{i\omega t}. \tag{11}$$

Equation (6) reveals (assuming $e^{i\omega t}$ time dependence)

$$\mathbf{E}_t = \pm\frac{iE_0k_p}{\gamma_{mnp}^2}J_m'(\gamma_{mnp}\rho)e^{\pm im\phi}\boldsymbol{\rho} - \frac{mE_0k_p}{\gamma_{mnp}^2\rho}J_m(\gamma_{mnp}\rho)e^{\pm im\phi}\boldsymbol{\phi}. \tag{12}$$

The transverse magnetic fields are found from Eq. (7)

$$\mathbf{H}_t = \frac{iE_0\epsilon_0\omega_{mnp}}{\gamma_{mnp}^2}J_m'(\gamma_{mnp}\rho)e^{\pm im\phi}\boldsymbol{\phi} \pm \frac{mE_0\epsilon_0\omega_{mnp}}{\gamma_{mnp}^2\rho}J_m(\gamma_{mnp}\rho)e^{\pm im\phi}\boldsymbol{\rho}. \tag{13}$$

The dispersion relation $\omega(k)$ for the TM_{mnp} modes can be found by manipulating the equation for γ, ω, and $k = p\pi/L$.

$$\gamma_{mnp}^2 = \mu\epsilon\omega_{mnp}^2 - k_p^2 \tag{14}$$

$$\omega_{mnp} = c\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{L^2}} \tag{15}$$

A. TM_{010} Modes

For the lowest TM mode $TM_{mnp} = TM_{010}$, the longitudinal component of the electric field is constant with respect to z and the transverse components vanish since $k_0 = 0$. Only the azimuthal magnetic field survives because $m = 0$. Notice also that for $p = 0$, $\omega_{010} = c\gamma_{010}$ and $c\epsilon_0 = 1/Z_0$ where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. The fields are

$$E_z = E_0 J_0(\gamma_{010}\rho)\ e^{i\omega t} \tag{16}$$

$$H_\phi = \frac{iE_0\epsilon_0\omega_{010}}{\gamma_{010}^2} J_0'(\gamma_{010}\rho) \tag{17}$$

$$= \frac{-iE_0}{Z_0} J_1(\gamma_{010}\rho). \tag{18}$$

The dispersion relation for several TM_{01p} modes in Fig. (1) is useful for later comparison to other linac geometries.

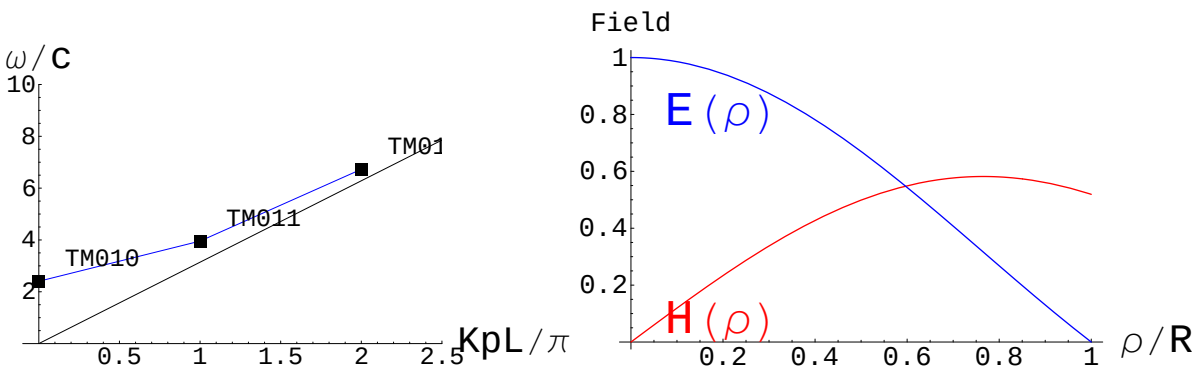

FIG. 1: The dispersion relation for the three lowest TM_{01p} modes of the uniform cylindrical cavity (*left*)[1]. The normalized electric field (*blue*) and magnetic field (*red*) for the lowest TM_{010} mode (*right*).

A drift tube linac (DTL) employing a standing TM_{010} wave in its resonant cavity is useful for particle acceleration. Figure (1) shows that charged particles at the center of the cavity ($\rho = 0$) feel a force *only* along the z-axis varying only in time. As long as particles are shielded by appropriately spaced drift tubes during the decelerating phase of the of the wave, they can be accelerated to energies much greater than those achieved with electrostatic machines. However, the energies of DTLs are limited by size of the machine. RF frequency on the order of GHz shortens the drift regions to less than a meter ($f\lambda = c$), but still doesn't ameliorate the size requirements for high energy DTLs: Even with large DTL *acceleration gradients* of 100 kV/m, particle energies of 50 GeV would require a 50 km DTL.

IV. TRAVELING WAVES IN WAVE GUIDES

A possible solution to the size limit of drift tube linacs is to use traveling waves synchronized with the particles' velocities. Rather than shielding the particle during the decelerating phase, the accelerating phase of the traveling wave follows the particle, and the particle feels a continuous positive field. The TM modes of a traveling wave in a circular cylindrical waveguide of radius $r = R$ are again most useful.

The fields for TM waves are determined from Maxwell's equations above, Eqs (6-8). The boundary conditions at $\rho = R$, and therefore the form of ψ, are the same. Since there are no endplates and no z boundary conditions, the z-dependence is properly expressed in Eq. (8). The non-zero field components for the lowest TM_{01} mode are

$$E_z = E_0 J_0(\gamma_{01}\rho)\ e^{i(kz-\omega t)} \tag{19}$$

$$E_\rho = -\frac{ikE_0}{\gamma_{01}} J_1(\gamma_{01}\rho)\ e^{i(kz-\omega t)} \tag{20}$$

$$H_\phi = -\frac{iE_0\epsilon_0\omega}{\gamma_{01}} J_1(\gamma_{01}\rho)\ e^{i(kz-\omega t)} \tag{21}$$

Just as in the case of a standing wave, a particle on axis will feel a force *only* in the z-direction. Additionally, if the particle is synchronized with the positive phase of the wave it will feel a force constant in time, see Sec. (IV A) below. However, the dispersion curve (Fig. (2)) and form of the phase velocity (Eq. (24)) for the TM_{01} mode reveal the obstacle in using traveling waves to accelerate synchronous particles:

$$\gamma_{01}^2 = (\frac{\omega}{c})^2 - k^2 \tag{22}$$

$$(\frac{\omega}{k})^2 = c^2(1 + \frac{\gamma_{01}}{k}^2) \tag{23}$$

$$v_p = c\sqrt{1 + \frac{x_{01}^2}{R^2k^2}}. \tag{24}$$

For a uniform cylindrical waveguide, the phase velocity of the wave is always greater than c preventing particle synchronization with the accelerating phase of the wave.

A. Synchronous Particles

Synchronous particles have velocity equal to the phase velocity of the wave. From Eq. (19), a particle at phase ϕ on axis in the waveguide feels an oscillating electric field

$$E_z = E_0 \sin(\omega t - kz + \phi). \tag{25}$$

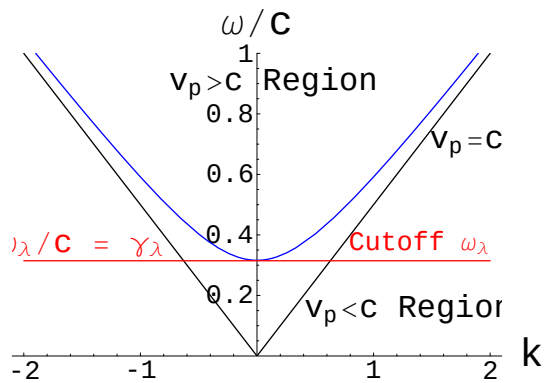

FIG. 2: The dispersion curve (*blue*) for the TM_{01} mode of the uniform circular cylindrical waveguide including plot of cutoff freqency ω_λ (*red*). The phase velocity is everywhere greater than the speed of any particle, $v_p > c$.

This wave has phase velocity $v_p = \omega/k$ and the particle's velocity is called $v(t) = z(t)/t$. Substituting into Eq. (25), one obtains

$$E_z \;=\; E_0 \sin[\omega t - (\frac{\omega}{v_p})(vt) + \phi]. \tag{26}$$

When $v_p = v$ the *synchronized* particle at phase ϕ_s feels a constant E-field

$$E_z \;=\; E_0 \sin(\phi_s). \tag{27}$$

If the velocities differ slightly, the field will have some small effective freqeuncy but will be nearly constant as felt by the particle. This will be explored in more detail, but for now since the TM traveling waves above have $v_p > c$, particle synchronization is precluded.

V. IRIS LOADING

The phase velocity of wave can be lowered by introducing periodic structure into the waveguide. An *iris-loaded* waveguide, Fig. (3), can be thought of as a waveguide fitted with equally-spaced, axially-bored plates or as a series of resonant cavities coupled by fields that penetrate the beam holes[5].

An analytic solution to Maxwell's equation for the iris-loaded waveguide is beyond the scope of this paper, but one can make conclusions about the dispersion and phase velocity of waves in this waveguide. Arguing that solutions to the Helmholtz equation, Eq. (9), must be z-periodic with frequency $2\pi/L$ to match the periodicity of the loaded waveguide,

FIG. 3: A cross section of the circular cylindrical iris-loaded waveguide with beam holes of radius A.

one can employ the *Floquet Theorem.* A paraphrasing of Floquet's Theorem states that the solution(s) to a differential equation with periodic coefficients are quasi-periodic in that their form differs from period to period by only a phase[5]. For the TM modes, one can write

$$E_z(\rho,\phi,z,t) \;=\; e^{-ik_0 z}E_z(\rho,\phi,z)e^{i\omega t} = phase \times periodic\ function, \tag{28}$$

where, because of Floquet's Theorem, $E_z(\rho,\phi,z+L) = E_z(\rho,\phi,z)$. The periodicity of E_z allows for the expansion in a Fourier series

$$E_z(\rho,\phi,z) \;=\; \sum_{q=-\infty}^{\infty} f_q(\rho,\phi)e^{-i\frac{2\pi q}{L}z}. \tag{29}$$

The functions f_q must still satisfy the wave equation and boundary conditions, so $f_q(\rho,\phi) = J_m(\gamma_{mnq}\rho)e^{\pm im\phi}$ as before. For the TM_{01} modes, the fields become

$$E_z \;=\; e^{-(iK_q z - i\omega t)} \sum_{q=-\infty}^{\infty} E_q J_0(\gamma_{01q}\rho) \tag{30}$$

$$E_\rho \;=\; ie^{-(iK_q z - i\omega t)} \sum_{q=-\infty}^{\infty} \frac{K_q E_q}{\gamma_{01q}} J_1(\gamma_{01q}\rho) \tag{31}$$

$$H_\phi \;=\; ie^{-(iK_q z - i\omega t)} \sum_{q=-\infty}^{\infty} \frac{\omega\epsilon_0 E_q}{\gamma_{01q}} J_1(\gamma_{01q}\rho) \tag{32}$$

where $K_q = k_0 + 2\pi q/L$, and $\gamma_{01q} = x_{01}/R$ are still the Bessel zeroes over the radius. As expected, particles on axis feel a force only in the z-direction.

One can think of the fields as the sum of q *space harmonics* each with wave number K_q. Taking $q = 0$ yields the *fundamental space harmonic* with wave number k_0[2]. The equations

for the frequency and phase velocity of each space harmonic are

$$\omega = c\sqrt{\gamma_{01q}^2 + K_q^2} = c\sqrt{\gamma_{01q}^2 + (k_0 + \frac{2\pi q}{L})^2} \tag{33}$$

$$v_{pq} = \frac{\omega}{K_q} = \frac{\omega}{k_0 + 2\pi q/L} = \frac{v_{p0}}{1 + 2\pi q/k_0 L}. \tag{34}$$

For large enough q, the qth space harmonic will have phase velocity well below the speed of light, and it can even be shown that the phase velocity of the fundamental harmonic has been brought less than c.

The dispersion curve can be constructed by making qualitative arguments about the complicated fields in the iris-loaded waveguide. Plotting $\omega(k_0)$ from Eq. (33) for $q = -1, 0, 1$ shows three typical dispersion curves shifted by $2\pi/L$. The curves for $q = -1, 1$ have significant portions below the diagonal where $v_p = c$. One might imagine, and it has been show experimentally that the dispersion relation is continuous and smoothly varying. The separate curves can be joined to form the periodic effective dispersion relation (*red*) in Fig. (4).

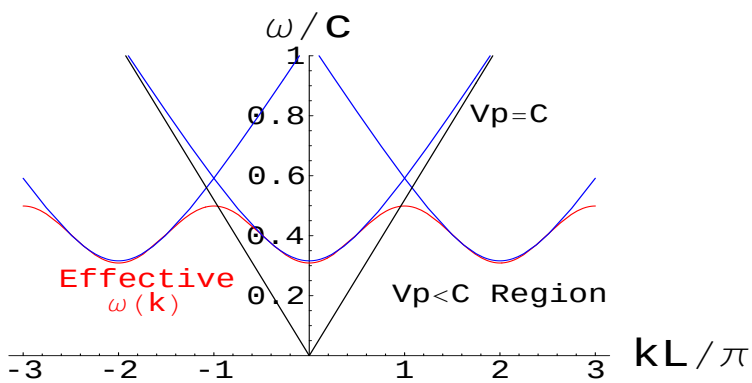

FIG. 4: The dispersion curve for the iris-loaded waveguide. In blue are the individual dispersion curves for the $q = -1, 0, 1$ space harmonics. In red is the experimentally confirmed effective dispersion curve for the iris-loaded waveguide.

This dispersion curve contains much information. Apparently, even the phase velocity of the fundamental space harmonic $q = 0$ has been modified. Though it is difficult to see in Fig. (4), the $q = 0$ harmonic (corresponding to $0 \leq k \leq \pi/L$) has a small portion beyond the $v_p = c$ line and therefore can be synchronized with particles. In practice, this fundamental harmonic is primarily used to accelerate particles at phase velocity $v_p \approx c$. The downward slopes of the effective dispersion relation, between odd and even multiples of π/L, are due

to negative values of k_0, waves traveling in the opposite direction, and are not used for acceleration. The dispersion curve has zero slope at integer multiples of π/L, and so the group velocity there vanishes $v_g = d\omega/dk = 0$; these are the wave numbers ($kL = 0, \pi, 2\pi...$) of standing waves in the L-periodic waveguide[5].

VI. PARTICLE DYNAMICS

It is never the case that particles are exactly synchronized with the waves, so it is important to understand how particles *close to* synchronization behave. Taking the energy, phase, and time of synchronous and non-synchronous particles to be U_s, $\phi_s = \omega t_s$, t_s, U, $\phi = \omega t$, and t, respectively; one can define $\Delta U = U - U_s$, $\Delta\phi = \phi - \phi_s$, and $\Delta t = t - t_s$. The energy of a synchronous electron propagating in the field from Eq. (27), $E_0 \sin\phi_s$, is related to that field as

$$-\frac{dU}{dz} = eE_0 \sin\phi_s \tag{35}$$

so one can relate the phases and energies of synchronous and non-synchronous particles[1]

$$\frac{d(\Delta U)}{dz} = eE_0[\sin(\phi_s + \Delta\phi) - \sin\phi_s]. \tag{36}$$

For a small phase difference $\Delta\phi$

$$\frac{d(\Delta U)}{dz} = eE_0[\sin(\phi_s)\cos(\Delta\phi) + \cos(\phi_s)\sin(\Delta\phi) - \sin\phi_s] \tag{37}$$

$$\approx eE_0 \Delta\phi \cos(\phi_s). \tag{38}$$

Assuming $v - v_s = \Delta v \ll 1$ and Taylor expanding in small Δv, the rate of change of the phase difference can be seen (note $\phi = \omega t$ above) to be

$$\frac{d(\Delta\phi)}{dz} = \omega(\frac{dt}{dz} - \frac{dt_s}{dz}) = \omega(v^{-1} - v_s^{-1}) = \omega((v_s + \Delta v)^{-1} - v_s^{-1}) \tag{39}$$

$$= -\omega\frac{\Delta v}{v_s^2}. \tag{40}$$

Using several easily proved relativistic relations,

$$\frac{dp}{dv} = \gamma^3 m, \tag{41}$$

$$\frac{dE}{dp} = mc^2\frac{dv}{dp}\frac{d\gamma}{dv} = mc^2(\frac{1}{\gamma^3 m})(\frac{\gamma^3 v}{c^2}) = v, \tag{42}$$

one can show that $\Delta v = \Delta U/\gamma^3 mv$, and therefore that

$$\frac{d(\Delta\phi)}{dz} = -\frac{\omega}{m\gamma^3 v_s^3}\Delta U. \tag{43}$$

Differentiating again with respect to z and using Eq. (38), one obtains a second order differential equation

$$\frac{d^2(\Delta\phi)}{dz^2} = -\frac{\omega}{m\gamma_s^3 v_s^3}\frac{d(\Delta U)}{dz} = -[\frac{eE_0\omega}{mc^3\gamma_s^3\beta_s^3}\cos(\phi_s)]\Delta\phi = -\kappa^2\Delta\phi. \tag{44}$$

This is just the equation for a simple harmonic oscillator. A particle with phase difference $\Delta\phi$ relative to a synchronous particle will oscillate in phase (and energy) about that synchronous particle provided $\kappa^2 > 0$. This constraint on κ demands that $\cos\phi_s > 0$ and that $-\frac{\pi}{2} < \phi_s < \frac{\pi}{2}$. Additionally, in Eq. (35) it is clear that for positive energy gain one needs $\sin\phi_s > 0$ and so

$$0 < \phi_s < \frac{\pi}{2}. \tag{45}$$

For high energies, $\gamma_s^3\beta_s^3$ in Eq. (44) is very large, and so κ^2 is very small. The particles oscillate very little about the synchronous phase, and the beam is a very tight bunch[1].

VII. PARTICLE CAPTURE AND BUNCHING

Along similar lines, one can investigate the conditions for the *capture* (into a synchronous accelerating phase) of particles injected into the beam with $v < c$. These particles and the accelerating wave with frequency ω travel distances differing by $dl = (c - v)dt$ in a time dt. This can be expressed as a difference in phase $dl = c/\omega\ d\phi$ implying[1]

$$\frac{d\phi}{dt} = \omega(1-\beta). \tag{46}$$

The particle feels a force dp/dt according to Eq. (35)

$$mc\,\frac{d(\gamma\beta)}{dt} = mc\,\frac{d}{dt}[\frac{\beta}{(1-\beta^2)^{\frac{1}{2}}}] = eE_0\sin\phi \tag{47}$$

Letting $\beta \equiv \cos x$ one can write

$$\frac{d}{dt}\cot x = \frac{eE_0}{mc}\sin\phi \tag{48}$$

$$\frac{1}{\sin^2 x}\frac{dx}{dt} = -\frac{eE_0}{mc}\sin\phi. \tag{49}$$

Using the chain rule, Eq. (46), and $(1-\beta) = 1 - \cos x$ this expression can be prepared for integration

$$\frac{1}{\sin^2 x}\,\frac{d\phi}{dt}\,\frac{dx}{d\phi} = -\frac{eE_0}{mc}\sin\phi \tag{50}$$

$$\frac{1-\cos x}{\sin^2 x}\,dx = -\frac{eE_0}{\omega mc}\sin\phi\; d\phi. \tag{51}$$

Integrating both sides from initial injection to final capture ($\beta_i \to \beta_c = 1$ and $\phi_i \to \phi_c$) and employing the trigonometric identity $tan\theta = \sqrt{(1-cos2\theta)/(1+cos2\theta)}$, one obtains the *capture condition*[1]

$$\cos\phi_c - \cos\phi_i = -\frac{\omega mc}{eE_0}\,[\frac{1-\beta_i}{1+\beta_i}]^{\frac{1}{2}}. \tag{52}$$

For a tuning of E_0, ω, and β_i such that the right hand side of Eq. (52) becomes -1, Figure (5) shows the relation of captured phase to injected phase $\phi_c = \arccos[\cos\phi_i - 1]$. Injection at $\phi_i = 0$ gives $\phi_c = \pi/2$ which according to Eq. (35) maximizes energy gain.

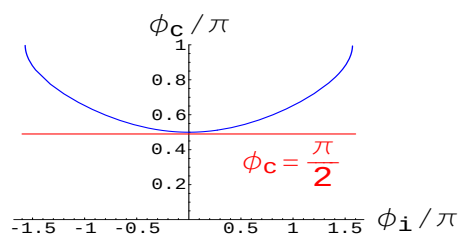

FIG. 5: Injected phase $-\pi/2 < \phi_i < \pi/2$ results in captured phase $\pi/2 < \phi_c < \pi$.

Because injection techniques are not perfect, understanding the behavior of particles injected *near*, but not quite at $\phi_i = 0$ is important. Taking $\phi_i = \delta\phi \ll 1$

$$\phi_c = \arccos[\cos(\delta\phi) - 1] \tag{53}$$

$$\approx \arccos[1 - \frac{(\delta\phi)^2}{2} - 1]\,\arccos[-\frac{(\delta\phi)^2}{2}] = \pi - \arccos[\frac{(\delta\phi)^2}{2}] \tag{54}$$

$$\approx \pi - [\frac{\pi}{2} - \frac{(\delta\phi)^2}{2}]. \tag{55}$$

The same process for $\phi_i = -\delta\phi$ yields a range of captured phase[1]

$$\frac{\pi}{2} \le \phi_c \le \frac{\pi}{2} + \frac{(\delta\phi)^2}{2} \tag{56}$$

for particles injected with a phase spread of $\delta\phi$ about $\phi_i = 0$. As hoped, in the capture process, particles injected with phase $\phi_i = \pm\delta\phi$ will be bunched even tighter about the ideal phase $\phi = \pi/2$.

VIII. CONCLUSIONS

Charged particles are accelerated using an electric field. Time-varying electric fields yield much higher particle energies than static fields. The use of traveling waves in a waveguide instead of standing waves in a cavity gives further increase in the energy of synchronized particles, because traveling waves continuously transfer energy to the particles. To attain synchronization the phase velocity of the traveling waves must be slowed by the use of a periodic, iris-loaded waveguide.

Within the range of synchronous phases $0 < \phi_s < \frac{\pi}{2}$, particles with phase close to ϕ_s will maintain stable orbits about the synchronous value. For a particular tuning of accelerator parameters, particles injected into the beam with velocity $v < c$ at phase $\phi_i \approx 0$ will be bunched very tightly about $\phi_s = \pi/2$ providing maximum acceleration.

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[1] S. Lee, *Accelerator Physics* (World Scientific, 1999).

[2] T. Wangler, *RF Linear Accelerators* (John Wiley and Sons, Inc., 1998).

[3] K. Wille, *The Physics of Particle Accelerators* (Oxford University Press, 2000).

[4] J. D. Jackson, *Classical Electrodynamics, Third Edition* (John Wiley and Sons, Inc., 1999).

[5] E. Wilson, *An Introduction to Particle Accelerators* (Oxford University Press, 2001).

[6] D. Edwards and M. Syphers, *An Introduction to the Physics of High Energy Accelerators* (John Wiley and Sons, Inc., 1993).