Notes on Anomalies

Nuno T. Leonardo

Massachusetts Institute of Technology

Abstract

It is introduced the concept of anomaly in a quantum field theory, resulting from non-invariance of the path-integral measure under a (classical) symmetry of the theory.

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1 Introduction

Given a symmetry of a Lagrangian – which is *classical* – there is no warranty that it may be elevated to a *quantum* symmetry, that is, the symmetry of the effective action. If the classical symmetry cannot be maintained in the process of quantization, the theory is said to have an **anomaly**.

To be precise, an anomaly will arise whenever extra (anomalous) terms appear in the Ward identities [1] due to the fact that the path integral measure is not invariant under that symmetry.

There are many types of anomalies; according to the symmetry under consideration: chiral, gauge, gravitational, supersymmetry.

Anomalies constitute an important feature of QFT with several crucial physical implications, both from the phenomenological and theoretical point of view.

They have been proved to play a fundamental role in many different contexts, as the decay of the neutral pion into two photons, the violation of lepton and baryon number in the Standard Model, the $U(1)_A$ problem.

Anomalies can also violate *gauge* symmetries, but in this case the theory becomes inconsistent, and therefore the cancellation of gauge anomalies is a fundamental requirement that any gauge theory should satisfy.

2 Axial anomaly in four dimensions

Consider the theory, described by the Lagrangian (density)

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + \mathcal{A}_\mu) - m]\psi$$

of a Dirac field ψ in 4-dimensional space-time coupled to an external gauge field $\mathcal{A}_\mu = \mathcal{A}_\mu^a T_a$; $\{T_a\}$ being generators of the gauge group G (compact, semi-simple, *e.g.* $SU(n)$).

The Lagrangian is invariant under the usual gauge transformations

$$\psi(x) \to g\psi(x) \qquad \mathcal{A}_\mu \to g[\mathcal{A}_\mu + \partial_\mu]g^{-1}$$

In the massless limit, $m \to 0$, it is also invariant under the global **chiral** symmetry

$$\psi(x) \to e^{i\gamma_5\alpha}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\gamma_5\alpha}$$

According to Noether's theorem, chiral invariance would imply the existence of a conserved axial-vector current,

$$j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$$

[1] Any symmetry in a QFT leads to the corresponding Ward Identities for the Green functions

which is gauge invariant.

The use of the classical equations of motion (Dirac equation) leads to the naive operator equation for the divergence of the axial current

$$\partial_\mu j_5^\mu + 2im\bar{\psi}\gamma_5\psi = 0$$

It is shown [Adler, Bell, Jackiw] that the chiral symmetry of $\mathcal{L}$ in the massless limit is destroyed at the quantum level. Indeed, the corresponding operator equation in the quantum theory receives a famous correction (the anomaly [2])

$$\partial_\mu j_5^\mu + 2im\bar{\psi}\gamma_5\psi = Q$$

where the topological charge Q is

$$Q \equiv \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\ tr\ \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}$$

(This is just the 2nd Chern character $ch_2(\mathcal{F})$, which is given in terms of the 3rd Chern-Simons form $Q_3(\mathcal{A},\mathcal{F})$ as $Q = \frac{1}{2}ch_2(\mathcal{F}) = \frac{1}{2}dQ_3(\mathcal{A},\mathcal{F})$, or in components

$$Q = \partial_\mu\ [\ \epsilon^{\mu\nu\rho\sigma}\ tr\ (\mathcal{A}_\nu\partial_\rho\mathcal{A}_\sigma\ +\ \frac{2}{3}\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma)]\ \ )$$

As a remark, for the general case of $d = 2n$ even dimensional space, the limit $m \to 0$ gives

$$\partial_\mu j_5^\mu = (-1)^{n+1}\frac{2}{n!(4n)^n}\epsilon^{\mu_1\mu_2\ldots\mu_{2n}}\mathcal{F}_{\mu_1\mu_2}\ldots\mathcal{F}_{\mu_{2n-1}\mu_{2n}}$$

Since Q is a total divergence, a 'new' axial current can be defined, $\tilde{j_5} \equiv j_5 - \frac{1}{2}Q_3(\mathcal{A},\mathcal{F})$, or in components

$$\tilde{j_5^\mu} \equiv j_5^\mu - [\ \epsilon^{\mu\nu\rho\sigma}\ (\mathcal{A}_\nu\partial_\rho\mathcal{A}_\sigma\ +\ \frac{2}{3}\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma)]$$

such that it is divergenceless, in the limit $m \to 0$,

$$d\tilde{j_5} = 0 \qquad (m = 0)$$

but is not gauge invariant, so cannot be interpreted as a physical current.

For the case of many fermions with masses m_i and chiral charges q_i interacting with the gauge potential $\mathcal{A}_{ij}^\mu$, the generalization of the operator equation considered above is

$$\partial_\mu J_5^\mu + 2im\bar{\psi}\gamma_5\psi = \frac{1}{2}\mathcal{Q}$$

with $\mathcal{Q}$ given by

$$\mathcal{Q} = \frac{1}{16\pi^2}\ tr\ q\mathcal{F}^2$$

[2] Since the current j_5^μ carry no group index, this is called an **Abelian anomaly**

3 Transformation of the measure

The anomaly is to be interpreted [3] as a symptom of the impossibility of defining a suitably invariant measure for integration over fermionic field variables. [4]

Consider the transformation of a column $\psi_n(x)$ of massless complex spin $\frac{1}{2}$ fermion fields interacting with a set of gauge fields $\mathcal{A}^\mu_\alpha(x)$

$$\psi(x) \mapsto g(x)\psi(x)$$

where $g(x)$ belongs to a proper representation of the gauge group.

Since these are fermionic variables, the measure transforms with the inverse of the determinant, as

$$[d\psi][d\bar{\psi}] \mapsto (\ det\ \hat{g}\ det\ \bar{\hat{g}}\ )^{-1}[d\psi][d\bar{\psi}]$$

where the operators $\hat{g}_{mn(x,y)} \equiv g(x)_{mn}\delta^4(x-y)$ and $\bar{\hat{g}}_{mn(x,y)} \equiv [\gamma_0 g(x)^\dagger_{mn}]_{mn}\delta^4(x-y)$, and $\bar{\psi} = \psi^\dagger\gamma_0$

In case $g(x)$ is a unitary non-chiral transformation, $g(x) = e^{i\alpha(x)t}$, with t an hermitian matrix not involving γ_5 and $\alpha(x)$ a real function, then $\hat{g}$ is pseudounitary, $\bar{\hat{g}}\hat{g} = 1$, and the measure is therefore left invariant.

In the case, instead, of a chiral transformation, $g(x) = e^{i\gamma_5\alpha(x)t}$, then $\hat{g}$ is pseudohermitian, $\bar{\hat{g}} = \hat{g}$, and the measure is not left invariant; rather $[d\psi][d\bar{\psi}] \mapsto (det\hat{g})^{-2}[d\psi][d\bar{\psi}]$. The transformation of the measure can be written

$$[d\psi][d\bar{\psi}] \mapsto e^{i\int d^4x\alpha(x)A(x)}[d\psi][d\bar{\psi}]$$

in terms of the anomaly function [5]

$$A(x) = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^a_{\mu\nu}\mathcal{F}^b_{\rho\sigma}tr\{t_a t_b t\}$$

This transformation for the measure has the same effect as if the Lagrangian density were not invariant, but instead

$$\mathcal{L}_{eff}(x) \mapsto \mathcal{L}_{eff}(x) + \alpha(x)A(x)$$

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[3] *à la* Fujikawa

[4] Fujikawa's analysis is based on an expansion of the fermionic variables of integration in eigenfunctions of the gauge invariant Dirac operator, and using path integrals in Euclidean space

[5] In the special case in which t is the unit matrix, $A(x)$ is known as the Chern-Pontrjagin density

4 Anomaly and index theorem

Consider again the Lagrangian

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu D_\mu - m]\psi$$

with $D_\mu = \partial_\mu + \mathcal{A}_\mu$. The corresponding fermionic functional integral is

$$Z = \int [d\psi][d\bar{\psi}] e^{i\int d^4x \bar{\psi}[i\gamma^\mu D_\mu - m]\psi}$$

Introducing the local transformations on the fermion fields

$$\psi(x) \to e^{i\alpha(x)\gamma_5}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\alpha(x)\gamma_5}$$

the corresponding transformation of the functional integral, combining the variation of the action and on the measure, gives

$$Z = \int [d\psi][d\bar{\psi}] e^{i\int d^4x[\bar{\psi}[i\gamma^\mu D_\mu - m]\psi + \alpha(x)\{\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) - 2im\bar{\psi}\gamma_5\psi + A(x)\}]}$$

where

$$A(x) = \frac{1}{16\pi^2} tr\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}$$

Varying the exponent in the expression for Z with respect to $\alpha(x)$ gives precisely the ABJ anomaly equation.

Consider now the eigenvalues and eigenfunctions of the Dirac operator $iD[\mathcal{A}] \equiv i\gamma^\mu D_\mu$,

$$iD[\mathcal{A}]\phi_n(x) = \lambda_n\phi_n(x)$$

with

$$\int d^4x \phi_m^\dagger(x)\phi_n(x) = \delta_{mn}$$

and express

$$\psi(x) = \Sigma a_n\phi_n(x), \qquad \bar{\psi}(x) = \Sigma\bar{b}_n\phi_n^\dagger(x)$$

where the coefficients $a_n, \bar{b}_n$ are elements of a Grassmann algebra.

The measure is then written as $[d\psi][d\bar{\psi}] = \Pi_{mn} da_n d\bar{b}_m$, and it transforms under the chiral transformation above as $[d\psi][d\bar{\psi}] \mapsto e^{i\int d^4x\alpha(x)A(x)}[d\psi][d\bar{\psi}]$, with $A(x)$ given in terms of the Dirac operator eigenfunctions as

$$A(x) = 2\Sigma_n\phi_n^\dagger(x)\gamma_5\phi_n(x)$$

Consider the Euclidean spacetime R^4 compactified to the sphere S^4, with the usual boundary condition of vanishing fields at infinity. [6]

The Atiya-Singer index theorem for the Dirac operator reads

$$ind\ iD_R[\mathcal{A}] \equiv n_+ - n_- = \frac{-1}{2(2\pi)^2}\int_{S^4} tr\mathcal{F}^2$$

where $iD_R \equiv iDP_R$ and $n_\pm$ are the number of zero modes ϕ_0 of iD with chirality equal to $\pm$, *i.e.* $\gamma_5\phi_0 = \pm\phi_0$. Notice the r.h.s. is simply the anomaly.

Now, in the expression for $A(x)$ the term $\sim \phi_n^\dagger\gamma_5\phi_n$ vanishes unless $\lambda_n = 0$; *i.e.*, only the zero modes of iD contribute to the sum. Indeed, note that $iD(\gamma_5\phi_n) = -\gamma_5\ iD\phi_n = -\lambda_n(\gamma_5\phi_n)$ and so $\gamma_5\phi_n$ is eigenvector with eigenvalue $-\lambda_n$, and therefore is orthogonal to ϕ_n unless $\lambda_n = 0$. Thus the expression for $A(x)$ gives simply the index of iD.

This shows then that the expression for the anomaly is directly related with [7] the index theorem.

As a remark, note that the non-Abelian gauge anomalies can also be related to the index of a six dimensional Dirac operator,in an analogous way. [8]

5 Non-perturbative (Witten) Anomaly

The non-perturbative anomaly reflects the impossibility to define the fermionic determinant for some theories in such a way that it is invariant under large gauge transformations (*i.e.*, those not connected with the identity).

This was introduced for the case of an $SU(2)$ theory coupled to an odd number of chiral fermion doublets, by Witten.

Consider the theory of an $SU(2)_L$ gauge field $\mathcal{A}_\mu$ and a fermion doublet ψ, described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{YM} + i\bar{\psi}_L\gamma^\mu D_\mu\psi_L$$

for which the calculation of the effective action involves

$$\int [d\psi_L][d\bar{\psi}_L]e^{-\int \bar{\psi}_L\gamma^\mu D_\mu\psi_L} = (detD)^{\frac{1}{2}}$$

where the square root of the determinant appears because the integration is only over the left-handed component.

The gauge transformations being maps $g(x) : S^4 \to SU(2)$, belong to the non-trivial homotopy group $\pi_4(SU(2)) \cong Z_2$. The problem then is that under large gauge transformations the square root of the determinant changes sign;

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[6] for the case $M = S^n$ the manifold curvature terms do not contribute to the operator index

[7] it can be seen as a local form of

[8] the index theorem there involves $tr\ \mathcal{F}^3 = dQ_5[\mathcal{A},\mathcal{F}]$, instead.

i.e., it is not possible to define it in an invariant way, and the theory therefore is not well defined.

The problem can be rephrased in an equivalent way by noting that whereas D has a well defined determinant, the operator of interest here, $D_L \equiv D\frac{1}{2}(1+\gamma_5)$ while gauge invariant also, does not have a well defined determinant, because it does not map the space of fermion fields of one handedness into itself, but rather into the space of fields of the other handedness.

Had we have an even number of fermion doublets, there would not be an anomaly, since each doublet would give a minus sign, and cancel out the anomaly.

Of course, in the Standard Model the even number of $SU(2)_L$ doublets (per generation) warranties that this anomaly is indeed not present.

6 Parity anomaly in odd-dimensions

In spaces of odd-dimensions no gauge anomalies are expected. This results from the fact that $SO(2l+1)$ has no complex representations.

However here another type of anomaly – **parity anomaly** – appears, in which the parity symmetry of the classical action is not maintained through quantization. [9]

Defining conveniently the transformation of the fields under a parity P, and using a convenient regularization method, it is shown that the effective action $W[\mathcal{A}]$ transforms under P as

$$W[\mathcal{A}] \to W[\,^{P}\mathcal{A}] = \bar{W}[\mathcal{A}]$$

where the bar denotes complex conjugation.

[9] The parity anomaly in $2l+1$ dimensions is related to the Abelian anomaly in $2l+2$ dimensions