Notes on Gauge Theory

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Abstract

An overview of the differential geometrical formulation of gauge fields is presented.

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1 Introduction

The natural framework for describing gauge fields is provided by the concept of fibre bundles. Indeed, a gauge potential can be regarded as a local expression for a connection in a principal bundle; the Yang-Mills field strength is then identified with the local form of the curvature associated with the connection.

This construction is fundamental in the study of anomalies, non-perturbative effects, instantons and vacuum structure in gauge theories.

2 Gauge principle

Let $P = (P, M, \pi, G)$ be a principal bundle over spacetime, with a connection w and curvature Ω; $\{\phi_i\}$ local trivializations, called local gauges; σ_i the section defined by the local gauge ϕ_i; $\mathcal{A}_i$ the gauge potential in the local gauge ϕ_i, $\mathcal{A}_i = \sigma_i^* \omega$; $\mathcal{F}_i$ the gauge field in the local gauge ϕ_i, $\mathcal{F}_i = \sigma_i^* \Omega$.

A gauge theory may be constructed in terms of a functional of the fields, the action [1] – the integral over spacetime of a Lagrange density or Lagrangian – which remains invariant under the action of a group, the gauge group. In constructing a Lagrangian invariant under (local) gauge transformations [2] we need to render the derivatives an invariant concept, corresponding to the use of covariant derivatives. What is needed is then a connection on a principal bundle over spacetime whose group is the gauge group. The pull-back of the connection one-form by a local section is a gauge potential and will in the quantum field theory give rise to gauge bosons. The pull-back of the corresponding curvature is a gauge field. Sections of associated vector bundles are called matter fields. When the coupling of the gauge potential with the matter fields is exclusively through the gauge covariant derivative it is called a minimal coupling.

3 Matter fields

Consider scalar [3] fields, denoted Higgs fields, and a representation $\rho : G \to GL(F)$ of the gauge group G on the vector space F – take F to be C^n and ρ a unitary representation of G on C^n.

A Higgs field of type (ρ, F) is a section ψ over a vector bundle with typical fibre F associated with P by the representation ρ.

In an equivalent way, a Higgs field of type (ρ, F) may be viewed as a map $\tilde{\psi} : P \to F$ which is invariant under G, *i.e.* $\tilde{\psi}(ug) = \rho(g^{-1})\tilde{\psi}(u), \;\; u \in P, \; g \in G$.

[1] and a measure in the functional space

[2] in addition to Lorentz transformations

[3] for which the transformation behaviour under spacetime coordinate changes is trivial

4 Gauge transformations

Consider spacetime a compact d−dimensional manifold M with Euclidean signature. In general this manifold will be covered by many patches which are topologically equivalent (homeomorphic) to R^d. Let G be a compact Lie group – usually taken to be $SU(N)$. Within this formalism the usual gauge fields are nothing but a map where to each point of M is assigned a one-form valued on the Lie algebra of G

$$x \mapsto \mathcal{A}(x) = T^a A^a_\mu(x) dx^\mu$$

where T^a are the hermitian group generators with the normalization $tr(T^a T^b) = \frac{1}{2}\delta^{ab}$. The definition of the gauge field is completed once we specify how it transforms from patch to patch. To have a properly defined gauge field in the whole manifold, gauge fields in the intersection of the patches need to be related by a gauge transformation g_{ij}

$$\mathcal{A}_j(x) = g^{-1}_{ij}(x)(\mathcal{A}_i(x) + d)g_{ij}(x)$$

These gauge transformations are called transition functions and the gauge field is called a connection on the principal bundle with base M and fibre G.

A gauge transformation corresponds to a change of local sections. It can also be taken as a global automorphism $\mathcal{F} : \pi^{-1}(x) \to \pi^{-1}(x)$ of the principal bundle.

4.1 Small and large gauge transformations

A gauge field is called trivial and the corresponding gauge transformation g_{ij} small whenever g_{ij} belongs to the trivial class.

Such small gauge transformations are the ones considered in perturbation theory.

Otherwise it is called large gauge transformation.

4.2 Pure gauge

Whenever $\mathcal{F}(x) = 0, \ \forall \ x$, the field is called a pure gauge field; then, in any coordinate neighborhood U, $\mathcal{F}$ can be written as

$$\mathcal{A}_\mu \to g \ \partial_\mu g^{-1}$$

where g is a map from the neighborhood U to the gauge group G.