Notes on θ-vacua & the Axion

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Abstract

Instantons can be interpreted as quantum mechanical tunneling phenomena in Yang-Mills gauge theories. They induce transitions between homotopically inequivalent vacua. The true ground-state of Yang-Mills theory then becomes a coherent mixture of all these vacuum states.

The so-called θ-angle defines the choice of the vacuum state among the possible physically distinct vacua. The inclusion of the θ- term in the Lagrangian, to account for the complicated vacuum structure of the theory, introduces P and T parity violation, unless $\theta = 0$. This is achieved introducing the axion.

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1 Introduction

The existence of topologically non-trivial gauge transformations means that the classical vacuum of non-Abelian gauge theories is not unique, but rather there is a degenerate set of vacua, each characterized by a topological winding number.

At the quantum level however, the physical vacuum – which is denoted θ-vacuum – is given by a superposition of those various classical vacua. Indeed, instantons are responsible for this quantum tunneling.

The effect of the θ-vacuum amounts to adding to the Yang-Mills action the so-called θ-term. The inclusion of this term in the QCD Lagrangian, which cannot be rotated away by a redefinition of the quark fields, implies violation of the symmetries P, T.

Physically equivalence of different θ-worlds can be achieved *e.g.* via the Peccei-Quinn mechanism. This is done by introducing an extra $U(1)$ symmetry, and implies the existence of a light pseudoscalar particle, the axion, which has itself a very rich phenomenology.

2 θ-vacuum

The minimum of the energy is reached for vanishing field strength at infinity, $\mathcal{F}_{\mu\nu} = 0$, which corresponds to pure gauge field configurations

$$\mathcal{A}_i = g^{-1}\partial_i g$$

Therefore, various vacuum configurations [1] correspond then to the many homotopy sectors and are characterized by their winding number. [2]

Denote by $|n>$ the local minima of the topological sector labeled by the integer n. These states are invariant under small, but not under non-trivial, gauge transformations. Actually, if $U(g_m)$ is an operator implementing, in the Hilbert space of the quantum theory, a gauge transformation g_m with winding number m, then

$$U(g_m)|n>=|n+m>$$

.

The physical vacuum, $|\theta>$, should however be invariant [3] under both small and large gauge transformations, and hence should be given by a linear combination of the $|n>$ states,

$$|\theta>=\sum_{n=-\infty}^{\infty} e^{in\theta}|n>$$

[1] between which the energy barrier is in fact finite

[2] $n \in \pi_3(SU(2)) = Z$

[3] up to a phase

To each value of θ corresponds a vacuum state. θ-vacuum states with different phases are actually orthogonal, and so the whole relevant part of Hilbert space corresponds to just one value of the phase θ – it is to be considered a parameter of the theory, since states corresponding to other θ values are completely decoupled.

3 The role of instantons: vacuum tunneling

The topologically distinct $|n>$ classical vacuum states result from the boundary condition that the gauge fields at the boundary of spacetime (infinity) are pure gauges. In particular this must be true for $t \to \pm\infty$. Accordingly, deform the spacetime boundary from S^3 to a cylinder, with the top and bottom surfaces corresponding to $t \to \pm\infty$.

Consider the gauge field configuration $\mathcal{A}_\mu(t,x) = g^{-1}\partial_\mu g$ which is pure gauge on the spacetime boundary characterized by the winding number q, interpolating between some gauge field $\mathcal{A}_i^n(x) = g_n^{-1}\partial_i g_n$ at time $t = -\infty$ with winding number n and some other pure gauge field $\mathcal{A}_i^m(x) = g_m^{-1}\partial_i g_m$ at time $t = \infty$ with winding number m.

Take the gauge $\mathcal{A}_0 = 0$. The winding number Q of g is given by

$$Q = \frac{1}{32\pi^2}\int_{S^4} tr\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} = m - n$$

Thus, tunneling from the state $|n>$ to $|m>$ is obtained by employing gauge fields with winding number $Q = m - n$.

Among such fields, a special subset that minimizes the Euclidean Yang-Mills action are the Q-instantons. [4]

Specially in the context of semiclassical calculations the path integral is restricted to solutions of the equations of motion, and hence instantons are the main contribution to tunneling processes.

4 The θ-term and the strong P and T problems

This complicated topological structure of the vacuum in non-Abelian theories is effectively taken into account by adding a term to the Yang-Mills action.

Consider the quantum transition amplitude between θ-vacuum states $|\theta>$, $|\theta'>$, by decomposing these in terms of $|n>$ *states* and the correspondent path integral expression, [5]

[4](anti-) self-dual fields that are solutions of the equations of motion

[5]in the second step is performed the change of variables $m \mapsto n + Q$, and the measure $[d\mathcal{A}^Q]$ denotes integration over gauge fields interpolating between $|m>$ and $|n>$

$$\begin{aligned}
<\theta|e^{-iHt}|\theta'> &= \Sigma_{m,n} e^{-im\theta'} e^{in\theta} < m|e^{-iHt}|n> \\
&= \Sigma_{m,n} e^{-i(Q+n)\theta'} e^{in\theta} \int [d\mathcal{A}^Q] e^{-S_{YM}[\mathcal{A}]} \\
&= 2\pi\delta(\theta-\theta') \Sigma_Q e^{iQ\theta} \int [d\mathcal{A}^Q] e^{-S_{YM}[\mathcal{A}]} \\
&= 2\pi\delta(\theta-\theta') \int [d\mathcal{A}] e^{-S_{YM}[\mathcal{A}]+iQ\theta}
\end{aligned}$$

Thus, in a θ-vacuum the action is altered by the addition of the term $S \mapsto S - iQ\theta$.

The inclusion of such a θ-term, with $\theta \neq 0$, would violate P and T symmetries.

Note that this θ-term is of the same form of the anomaly, thus it in principle could be absorbed with the anomalous Jacobian of an appropriate axial transformation of the quark fields.However, in QCD the existence of the mass matrix spoils this. Indeed, the redefinition of the fermion fields

$$\psi_f \mapsto e^{i\gamma_5 \alpha_f} \psi_f$$

where f is here denoting the flavour index, induces the change in the mass parameters

$$M_f \mapsto e^{2i\alpha_f} M_f$$

The physics cannot depend on a change of path integral variables, and observable quantities can depend only on the combination

$$e^{-i\theta} \Pi_f M_f$$

In particular, the fermion fields can always be defined such that $\theta = 0$, but then the imaginary fases so introduced in the mass parameters M_f again would account for P and T non-conservation. Conversely, we can rotate to the physical basis [6] by shifting the θ parameter to the effective $\bar{\theta}$

$$\bar{\theta} \equiv \theta + arg(detM)$$

Also, would any of the quark masses [7] be zero, and the θ-angle would have no effect.

A non-zero theta angle would give rise to a dipole moment for the neutron proportional to $|\bar{\theta}|$. The present experimental bound [8] predicts a value of the order $|\bar{\theta}| < 10^{-10}$.

Therefore, given such a small value for $|\bar{\theta}|$, we're left with the extreme fine tuning necessary to cancel these two apparently unrelated quantities — on the

[6] characterized by $M_{phys} = diag(m_u, m_d, m_s, \ldots)$

[7] as the experimentally non-favoured $m_u = 0$

[8] the estimated theoretical value for the neutron dipole moment depends on the models of nuclear structure

one hand the θ parameter, originating from the QCD vacuum, and on the other hand $arg(detM)$, where the mass matrix M results from the Yukawa couplings coming from the spontaneous symmetry breaking mechanism of the Standard Model. [9] This fine tuning is known as the **strong CP problem**.

5 Axions

In order to explain why θ is so small, Peccei and Quinn proposed that θ become a dynamical variable, which can relax to a minimum of the effective potential at which P and T are conserved. Here the different θ's do not describe different theories, but merely distinguishes different vacuum states with different vacuum energies in a given theory. Had been proven that $\bar{\theta} = 0$ corresponds indeed to the true vacuum, then $\bar{\theta}$ will roll down to 0 regardless of the initial value of $\bar{\theta}$.

Their idea was taken up by F. Wilczek and S. Weinberg, who noted then the necessity of a light spinless particle, the Peccei-Quinn-Weinberg-Wilczek (PQWW) *axion*.

The axion of the original Peccei-Quinn model has been ruled out by experiment. But the $U(1)_{PQ}$ symmetry breaking scale of the model can be shifted to higher energy scales making the axion more or less invisible.

The axion theory includes some $U(1)$ symmetry that is spontaneously broken at energies much higher than those associated with QCD, and is also broken by an anomaly involving the gluon fields.

The axion can actually arise in various forms: as the phase of the Higgs field, as a composite object, or as a degree in effective theories descending from the remnants of a more fundamental theory.

An excellent review of many phenomenological aspects of the axion is presented by elsewhere. [10] It includes as well a discussion of the axion in higher dimensional theories, including the phenomenology of superstring axions.

A review of the experimental searches for the invisible axion is presented elswhere. [11] Includes cavity experiments, based on the Primakov conversion of the axion to microwave photons in a high Q-resonator permeated by an intense magnetic field; cosmological production and stellar evolution constraints.

[9] experimentally very poorly understood
[10] Phys.Rept.150:1-177,1987
[11] Phys.Rept. 325:1-39,2000