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Mathematical Preliminaries

Fourier Transforms.

We shall move freely between the momentum and the position space representations of quantum operators so you should be comfortable in taking Fourier tranforms. For a function $f(\vec{x})$, its Fourier transform $\tilde{f}(\vec{k})$ is given by

$$f(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, \tilde{f}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$
$$\tilde{f}(\vec{k}) = \int d\vec{x}\, f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}.$$

Aside from expanding the quantum field in terms of creation and annihilation operators, the Fourier pair that you will see most often is that of the δ-function and the constant function:

$$\int d\vec{k}\, \delta^3(\vec{k}) e^{i\vec{k}\cdot\vec{x}} = 1$$
$$\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k}).$$

Complex Analysis.

You should be familiar with the basic properties of complex functions—analytic functions, contour integration, residues, poles, branch cuts, *etc.* For example, the residue theorem states

Theorem. Let Γ be a positively oriented simple closed contour within and on which a function is analytic except for a finite number of singular points z_1, z_2, ..., z_n interior to Γ. If B_1, B_2, ..., B_n denote the residues of f at those respective points, then

$$\int_\Gamma f(z)\, dz = 2\pi i (B_1 + B_2 + \cdots + B_n). \qquad \text{[see R. Churchill]}$$

The **residue** of a complex function $f(z)$ with an isolated singular point at z_0 is the coefficient b_1 of the Laurent series expansion of $f(z)$ about z_0,

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \qquad (0 < |z - z_0| < R).$$

You should review a fairly elementary book on complex analysis if you are not comfortable with these terms.

Representation Theory.

The symmetries that exist in nature provide a strong constraint on the physical threories that we study. The set of symmetry transformations form a group $\mathcal{G}$; for example, $\mathcal{G}$ may correspond to the translations of space-time or to Lorentz transformations ($SO(3,1)$). A representation is a mapping of an element of a group $g \in \mathcal{G}$ to an operator $U(g)$ that acts on our Hilbert space of states,

$$U : g \to U(g),$$

such that it preserves the group multiplication law:

$$U(g_1)U(g_2) = U(g_1 g_2).$$

In a quantum theory, these operators must conserve probabilities, so that the representation must be **unitary**‡:

$$U^\dagger(g)U(g) = U(g)U^\dagger(g) = 1.$$

The representation should map the identity element e of the group ($eg = ge = g$) to the identity operator:

$$U(e) = 1.$$

These are the basic properties shared by all the unitary representations that we shall encounter in this course. For specific groups such as the Lorentz group, $\mathcal{G} = SO(3,1)$, we impose further restrictions on the action of $U(g)$ on the states.

Constructing The Quantum Field — First Week

A Single Free Spinless Particle of Mass μ.

The states of the first quantum field theory that we shall study are eigenvectors of the momentum operator and are normalized as follows:

$$\vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$$
$$\langle\vec{k}|\vec{k}'\rangle = \delta^3(\vec{k} - \vec{k}')$$

‡ Since the probabilities and not the amplitudes must be preserved, anti-unitary operators are allowed as well.

This set of states forms a complete basis satisfying the orthogonality relation,

$$\langle\vec{k}|\vec{k}'\rangle = \delta^3(\vec{k}-\vec{k}')$$
$$1 = \int d^3\vec{k}\,|\vec{k}\rangle\langle\vec{k}|$$

For this theory to describe adequately a *relativistic* particle, it must incorporate the symmetries of spacetime: rotational invariance and Lorentz invariance. This means that these symmetry groups must be represented by a set of unitary (to preserve probabilities) operators that act on our basis. For example, if R is a rotation matrix, there should be an operator $U(R)$ that rotates the states:

1. Unitarity: $U(R)U^\dagger(R) = 1$.
2. Identity: $U(1) = 1$.
3. Preserves the multiplication law: $U(R_1)U(R_2) = U(R_1R_2)$.
4. Rotates vectors: $|\psi'\rangle = U(R)|\psi\rangle \quad \rightarrow \quad \langle\psi'|\vec{P}|\psi'\rangle = R\langle\psi|\vec{P}|\psi\rangle$.
5. Preserves the Hamiltonian: $U^\dagger(R)HU(R) = H$.

The above theory is not invariant under Lorentz boosts as might be expected since $|\vec{k}\rangle$ only contains the spatial coordinates. A theory which is invariant under the full Lorentz group ($SO(3,1)$) normalizes the states and the measure of momentum space as follows:

$$|k\rangle = (2\pi)^{3/2}(2\omega_{\vec{k}})^{1/2}|\vec{k}\rangle \text{ with } \omega_{\vec{k}} \equiv k_0 = \sqrt{\vec{k}^2+\mu^2}, \qquad \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})}.$$

The orthogonality and completeness statements become

$$\langle k|k'\rangle = (2\pi)^3(2\omega_{\vec{k}})\delta^3(\vec{k}-\vec{k}')$$
$$1 = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})}|k\rangle\langle k|.$$

If we define the action of applying a Lorentz tranformation on a state to be $U(\Lambda)|k\rangle = |\Lambda k\rangle$, then $U(\Lambda)$ satisfies all the requisite properties:

1. Unitarity: $U(\Lambda)U^\dagger(\Lambda) = 1$.
2. Identity: $U(1) = 1$.
3. Preserves the multiplication law: $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$.
4. 4-Vectors transform correctly: $U^\dagger(\Lambda)PU(\Lambda) = \Lambda P$.

Thus far the construction of the theory has proceeded without any appearance of the multi-particle nature of quantum field theory. However, if we introduce a position operator, $\vec{X}$, we discover that the particle has a non-zero probability of being outside of the forward light cone:

$$\langle\vec{X}|e^{-iHt}|\vec{X}=\vec{0}\rangle = -\frac{e^{-\mu r}}{2\pi^2 r}\int_\mu^\infty z dz\, e^{-(z-\mu)r}\sinh(\sqrt{z^2-\mu^2}\,t) \neq 0.$$

This single particle theory violates **causality**. Only by developing a multi-particle theory can causality be restored.

A Theory Free Spinless Particles of Mass μ.

The basis of states, called the Fock space, for a multiparticle theory contains a vacuum state $|0\rangle$, single particle states $|\vec{k}\rangle$, two particle states $|\vec{k}_1,\vec{k}_2\rangle$ $(=|\vec{k}_2,\vec{k}_1\rangle)$, *etc.* These states satisfy:

$$\begin{array}{llll}
\langle 0|0\rangle = 1 & \langle\vec{k}|\vec{k}'\rangle = \delta^3(\vec{k}-\vec{k}') & \langle\vec{k}_1,\vec{k}_2|\vec{k}_1',\vec{k}_2'\rangle = \delta^3(\vec{k}_1-\vec{k}_1')\delta^3(\vec{k}_2-\vec{k}_2') & \\
 & & \qquad +\delta^3(\vec{k}_1-\vec{k}_2')\delta^3(\vec{k}_2-\vec{k}_1') & \\
H|0\rangle = 0 & H|\vec{k}\rangle = \omega_{\vec{k}}|\vec{k}\rangle & H|\vec{k}_1',\vec{k}_2'\rangle = (\omega_{\vec{k}_1}+\omega_{\vec{k}_2})|\vec{k}_1',\vec{k}_2'\rangle & \\
\vec{P}|0\rangle = 0 & \vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle & \vec{P}|\vec{k}_1',\vec{k}_2'\rangle = (\vec{k}_1+\vec{k}_2)|\vec{k}_1',\vec{k}_2'\rangle &
\end{array}$$

and the completeness relation becomes

$$1 = |0\rangle\langle 0| + \int d\vec{k}\,|\vec{k}\rangle\langle\vec{k}| + \frac{1}{2!}\iint d\vec{k}_1 d\vec{k}_2\,|\vec{k}_1,\vec{k}_2\rangle\langle\vec{k}_1,\vec{k}_2| + \cdots.$$

In studying the multiparticle theory, it is convenient to introduce creation and annilation operators. For example, let the operator $a^\dagger_{\vec{k}}$ ($a_{\vec{k}}$) create (annihilate) a meson with momentum $\vec{k}$: *e.g.*

$$a^\dagger_{\vec{k}}|0\rangle = |\vec{k}\rangle \qquad \text{and} \qquad a_{\vec{k}}|\vec{k}\rangle = |0\rangle.$$

If these creation and annihilation operators satify the usual commutation reltions,

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}') \qquad [a_{\vec{k}}, a_{\vec{k}'}] = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0$$
$$H = \int d^3\vec{k}\,\omega_{\vec{k}}a^\dagger_{\vec{k}}a_{\vec{k}} \qquad \vec{P} = \int d^3\vec{k}\,\vec{k}a^\dagger_{\vec{k}}a_{\vec{k}},$$

the above eigenvalue equations for the energy and the momentum as well as the normalizations of the states are reproduced.

We have again begun with a non-relativistic theory which we would like to modify to satisfy the principles of relativity. One of these principles is that of **local commutativity**, that operators with spacelike separations must commute. Thus observable operators must necessarily be fields which depend on each space-time point. Thus let $\phi^a(x)$ $[a = 1,\ldots,N]$ represent a set of quantum fields (observable operators). They should satisfy the following properties:

1. Local Commutativity: $[\phi^a(x),\phi^b(y)] = 0$ for $(x-y)^2 < 0$.
2. Hermitian Operators: $\phi^a(x) = \phi^{a\dagger}(x)$.
3. Translations: $e^{-iP\cdot a}\phi^a(x)e^{iP\cdot a} = \phi^a(x-a)$.

4. Lorentz Transfomations: $U^\dagger(\Lambda)\phi^a(x)U(\Lambda) = \phi^a(\Lambda^{-1}x)$.
5. Linearity: $\phi^a(x) = \int d^3\vec{k}\,\left[F^a_{\vec{k}}(x)\,a_{\vec{k}} + G^a_{\vec{k}}(x)\,a^\dagger_{\vec{k}}\right]$.

This last condition is actually a simpifying assumption rather than a requirement for the theory. The form of the field consistent with these properties that emerges is

$$\phi(a) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}(2\omega_{\vec{k}})^{1/2}}\left[a_{\vec{k}}e^{-ik\cdot x} + a^\dagger_{\vec{k}}e^{ik\cdot x}\right].$$

This field satisfies the **Klein-Gordon equation**,

$$(\partial_\mu\partial^\mu + \mu^2)\phi(x) = 0,$$

and obeys the following commutation relation,

$$[\phi(x),\phi(y)] = \Delta_+(x-y) - \Delta_-(y-x)$$
$$\Delta_+(x) \equiv \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})}e^{-ik\cdot x}.$$

The Quantum Field

The quantum field was first constructed from the creation and annihilation operators but we could alternatively use the field itself to find $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$. If $\phi(x)$ satisfies

1. Hermiticity: $\phi(x) = \phi^\dagger(x)$
2. Klein-Gordon Equation: $(\partial_\mu\partial^\mu + \mu^2)\phi(x) = 0$

3′. Canonical Quantization:

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0$$
$$[\partial_0\phi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta^3(\vec{x}-\vec{y})$$

4. Tranforms Correctly under $SO(3,1)$ and translations: $U^\dagger(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ and $U^\dagger(a)\phi(x)U(a) = \phi(x-a)$
5. Completeness: $\forall_x\ [A,\phi(x)] = 0 \Rightarrow A \propto I$,

then $a_{\vec{k}}$ is

$$a_{\vec{k}} = \frac{\sqrt{2\omega_{\vec{k}}}}{2}\int \frac{d^3\vec{x}}{(2\pi)^{3/2}}\left(\phi(0,\vec{x}) + \frac{i}{\omega_{\vec{k}}}\dot{\phi}(0,\vec{x})\right)e^{-i\vec{k}\cdot\vec{x}}.$$

Canonical Quantization

In the canonical quantization of quantum mechanics, the classical position and conjugate momentum of classical mechanics are replaced by operators that satisfy the commutation relations,

$$[p_a(t),q^b(t)] = -i\delta^b_a \qquad [q^a(t),q^b(t)] = [p_a(t),p_b(t)] = 0.$$

Similarly, we can obtain *quantum field theory* through the analogous quantization of classical field theory. The fundamental physics of the theory determine the Lagrangian density:

$$\mathcal{L}(\phi^a(x),\partial_\mu\phi^a(x),x) \qquad S = \int d^4x\,\mathcal{L}.$$

and the Euler-Lagrange equations yield the equations of motion,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} - \frac{\partial\mathcal{L}}{\partial_\mu\phi^a} \equiv \partial_\mu\Pi^\mu_a - \frac{\partial\mathcal{L}}{\partial_\mu\phi^a} = 0.$$

As a example, for a theory with a quadratic Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2,$$

the equation of motion is the Klein-Gordon equation:

$$\partial_\mu\partial^\mu\phi + \mu^2\phi = 0.$$

The prescription for the canonical quantization of the theory is to replace the fields $\phi^a(x)$ and $\Pi^0_a(x) \equiv \partial\mathcal{L}/\partial(\partial_0\phi^a)$ by quantum fields that obey the equal-time commutation relations:

$$[\Pi^0_a(t,\vec{x}),\phi^b(t,\vec{y})] = -i\delta^b_a\delta^3(\vec{x}-\vec{y})$$
$$[\phi^a(t,\vec{x}),\phi^b(t,\vec{y})] = [\Pi^0_a(t,\vec{x}),\Pi^0_b(t,\vec{y})] = 0.$$

Symmetries and Conservation Laws — Second Week

Quantum Field Theory

The Hamiltonian of a free scalar field is given by

$$H = \int d^3\vec{x}\,\mathcal{H} = \int d^3\vec{x}\,\frac{1}{2}\left[(\Pi^0)^2 + (\vec{\nabla}\phi)^2 + \mu^2\phi^2\right]$$

which becomes upon the insertion of the form of the quantum field $\phi(x)$

$$H = \frac{1}{2}\int d^3\vec{k}\,\omega_{\vec{k}}\left(a_{\vec{k}}a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}}a_{\vec{k}}\right).$$

This Hamiltonian almost agrees with the form found from a rigorous development of the Hamiltonian in the Fock space. The difference is an infinite constant:

$$H = \int d^3\vec{k}\,\omega_{\vec{k}}\left(a^\dagger_{\vec{k}}a_{\vec{k}} + \tfrac{1}{2}\delta^3(0)\right).$$

The infinite constant is an artifact of the ordering ambiguity in quantizing a classical theory; however, it does create difficulties when we wish to include gravity for the gravitational field is effected by the presence of a vacuum energy density through a term containing the cosmological constant, λ:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} - \lambda g_{\mu\nu}).$$

To avoid such difficulties, we introduce **normal ordering** which eliminates the zero-point energy. The normal ordering prescription is as follows:

Normal Order: Given a set of free fields, the normal ordered product, denoted as $:\phi_1(x_1)\cdots\phi_n(x_n):$, is the usual product with all of the creation operators to the left and all of the annihilation operators to the right. If we define H to be $:H:$, then we recover our original Fock space Hamiltonian.

Symmetries and Conservation Laws

A symmetry is a transformation that maps one allowed motion of a system to another allowed motion. If the symmetry is continuous and acts on the generalized coordinates of some Lagrangian $L(q^a, \dot{q}^a, t)$ so that $q^a(t) \to q(t,\lambda)$, then it is possible to study the effects of an infinitesimal tranformation:

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial\lambda}\right|_{\lambda=0}.$$

This definition allows us to more precisely state when a transformation is a symmetry in terms of its effects on the Lagrangian.

Definition: A transformation is a symmetry if and only if $DL = dF/dt$ for some $F(q^a, \dot{q}^a, t)$ for aribitrary $q^a(t)$.

Noether's Theorem: For every symmetry there exists a conserved quantity Q given by

$$Q = \sum_a p_a Dq^a - F \qquad \frac{dQ}{dt} = 0.$$

Some familiar examples are

1. Translations: $\vec{r}_a \to \vec{r}_a + \lambda\hat{e}_a$ with $D\vec{r}_a = \hat{e}$ and $DL = 0 \Rightarrow$ the linear momentum is conserved, $\vec{P} = \sum_a m_a\dot{\vec{r}}_a$.
2. Time Translation: $\vec{r}_a(t) \to \vec{r}_a(t+\lambda)$ with $D\vec{r}_a = \dot{\vec{r}}_a$ and $DL = \dot{L} \Rightarrow$ the Hamiltonian (energy) is conserved, $H = \sum_a p_a\dot{q}^a - L$.
3. Rotations: $\vec{r}_a \to R(\lambda\hat{e})\vec{r}_a$ with $D\vec{r}_a = \hat{e}\times\vec{r}_a$ and $DL = 0 \Rightarrow$ the angular momentum is conserved, $\vec{J} = \sum_a \vec{r}_a \times \vec{p}_a$.

Symmtries in Classical Field Theory

One of the ways in which a classical field theory differs from a particle theory is with the concept of local charge conservation: any net change in some conserved charge must be accounted for by a net current flux into a region. Let us generalize the concepts above to those for a field theory. Let us study the continuous transformation $\phi^a(x) \to \phi^a(x,\lambda)$ for a system described by the Lagrange density $\mathcal{L}(\phi^a, \partial_\mu\phi^a, x)$. The infinitesimal version of this transformation is given by

$$D\phi^a = \left.\frac{\partial\phi^a}{\partial\lambda}\right|_{\lambda=0}.$$

Definition: A transformation is a symmetry if and only if $D\mathcal{L} = \partial_\mu F^\mu$ for some $F^\mu(\phi^a, \partial\phi^a, x)$ for aribitrary $\phi^a(x)$.

Noether's Theorem: For every symmetry there exists a conserved current given by

$$J^\mu = \sum_a \Pi^\mu_a D\phi^a - F^\mu \qquad \partial_\mu J^\mu = 0.$$

Note that J^μ can be shifted by an arbitrary χ^μ that satisfies $\partial_\mu\chi^\mu$: $J^\mu \to J^\mu + \chi^\mu$.

Space-time Translations

If a Lagrangian does not explicitly depend upon the space-time position, then the conserved currents corresponding to translations $\phi^a(x) \to \phi(x+\lambda e)$ lead to the energy momentum tensor:

$$T^{\mu\nu} = \sum_a \Pi^\mu_a \partial^\nu \phi^a - g^{\mu\nu}\mathcal{L}$$

which satisfies $\partial_\mu T^{\mu\nu} = 0$. The components of this tensor correspond to familiar quantities:

$$\begin{aligned} T^{00} &= \sum_a \Pi_a \dot{\phi}^a - \mathcal{L} = \mathcal{H} \\ T^{0i} &= \text{the density of the } i\text{th component of the momentum} \\ T^{ji} &= \text{the } j\text{th component of the current of the } i\text{th} \\ &\quad\ \text{component of the momentum.} \end{aligned}$$

We can also study a Lagrangian $\mathcal{L}$ invariant under Lorentz transformations, $a^\mu \to \Lambda^\mu_\nu a^\nu$. Under a Lorentz transformation, the fields transform as $\phi^a(x) \to \phi^a(\Lambda^{-1}(\lambda)x)$. If $\mathcal{L}$ only depends on x through the fields, then we can form the following conserved tensor:

$$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}$$

which is anti-symmetric in $\nu\lambda$ and satisfies $\partial_\mu M^{\mu\nu\lambda} = 0$. The six conserved charges in this theory are

$$J^{\mu\nu} = \int d^3\vec{x}\, M^{0\mu\nu} = \int d^3\vec{x}\, \left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right).$$

For example, the third component of angular momentum is given by

$$J^3 = J^{12} = \int d^3\vec{x}\, \left(x^1 T^{02} - x^2 T^{01}\right).$$

The three quantities $J^{0i} = \int d^3\vec{x}\,(x^0 T^{0i} - x^i T^{00})$ represent the motion of the center of energy. It is to Lorentz boosts as the angular momentum is to rotations.

Internal Symmetries

In addition to symmetries that correspond to transformations of space-time, other symmetries which act directly on the fields themselves rather than

their dependence on x are called **internal symmtries**. A first example is provided by the theory

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^1\partial^\mu\phi^1 + \tfrac{1}{2}\partial_\mu\phi^2\partial^\mu\phi^2 + \tfrac{1}{2}\mu^2(\phi^1)^2 + \tfrac{1}{2}\mu^2(\phi^2)^2 - g\left((\phi^1)^2 + (\phi^2)^2\right)^2.$$

This theory is invariant under a rotation of the fields

$$\begin{aligned} \phi^1 &\to \cos\lambda\,\phi^1 + \sin\lambda\,\phi^2 \\ \phi^2 &\to -\sin\lambda\,\phi^1 + \cos\lambda\,\phi^2. \end{aligned}$$

The conserved charge is thus $Q = \int d^3\vec{x}(\dot{\phi}^1\phi^2 - \dot{\phi}^2\phi^2)$; in terms of the creation and annihilation operators, Q becomes

$$Q = i\int d^3\vec{k}\left[a^{1\dagger}_{\vec{k}}a^2_{\vec{k}} - a^{2\dagger}_{\vec{k}}a^1_{\vec{k}}\right].$$

The interpretation of this charge becomes more clear if we introduce a complex field ψ given by

$$\begin{aligned} \psi &= \frac{\phi^1 + i\phi^2}{\sqrt{2}} = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}(2\omega_{\vec{k}})}\left[b_{\vec{k}}e^{-ik\cdot x} + c^\dagger_{\vec{k}}e^{ik\cdot x}\right] \\ \psi^\dagger &= \frac{\phi^1 - i\phi^2}{\sqrt{2}} = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}(2\omega_{\vec{k}})}\left[c_{\vec{k}}e^{-ik\cdot x} + b^\dagger_{\vec{k}}e^{ik\cdot x}\right] \end{aligned}$$

where

$$b_{\vec{k}} = \frac{a^1_{\vec{k}} + ia^2_{\vec{k}}}{\sqrt{2}} \qquad c_{\vec{k}} = \frac{a^1_{\vec{k}} - ia^2_{\vec{k}}}{\sqrt{2}}.$$

Then Q becomes the difference in the number of particles and antiparticles:

$$Q = \int d^3\vec{k}\left(b^\dagger_{\vec{k}}b_{\vec{k}} - c^\dagger_{\vec{k}}c_{\vec{k}}\right) = N_b - N_c.$$

The free field part of the Lagrangian in terms of this complex field is

$$\mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - \mu^2\psi^\dagger\psi.$$

Discrete Symmetries

In addition to the continuous symmetries discussed last week, another important class of symmetries are discrete symmetries which act as follows: $q^a(t) \to q^{a\prime}(t)$ and $L \to L$. With the exception of time-reversal, all the discrete symmetries are implemented by a unitary operator which preserves the Hamiltonian:

$$U^\dagger q^a U = q^{a\prime} \qquad U^\dagger H U = H.$$

Example 1. $\phi(x) \to -\phi(x)$

This transformation is a symmetry of any Lagrangian whose terms only contain even powers of ϕ, *e.g.*

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 - g\phi^4.$$

For the free field theory case ($g = 0$), the unitary operator for this symmetry is $U = (-1)^N$, where N is the meson number. Even for interacting fields, this symmetry means that no process can convert an even number of mesons to an odd number.

Example 2. Charge Conjugation

The $SO(2) = U(1)$ symmetric Lagrangian,

$$\mathcal{L} = \sum_{a=1}^{2}\left(\tfrac{1}{2}(\partial_\mu\phi^a)^2 - \tfrac{1}{2}\mu^2(\phi^a)^2\right) - g\left(\sum_{a=1}^{2}(\phi^a)^2\right),$$

actually has an $O(2)$ symmetry as it is invariant under the transformation $\phi^1 \to \phi^1$, $\phi^2 \to -\phi^2$. In the free field case, the unitary operator that produces this transformation is $U_C = (-1)^{N_2}$ where N_2 is the number of the ϕ^2 mesons. This operator acts as a conjugation on the $\psi = (\phi^1 + i\phi^2)/\sqrt{2}$ complex field:

$$U_C^\dagger \psi U_C = \psi^\dagger \qquad U_C^\dagger \psi^\dagger U_C = \psi.$$

Parity Transformations

A parity transformation is an improper rotation which in three dimensions can be accomplished by sending $\vec{x} \to -\vec{x}$. The the action of this transformation could in general mix the fields as follows:

$$\phi^a(t,\vec{x}) \to \sum_{b=1}^{n} M^{ab}\phi^b(t,-\vec{x}).$$

We shall discuss this symmetry through several examples.

Example 1. Scalars and Pseudoscalars

The free field Lagrangian, $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2$, acually has two types of parity symmetries:

$$P : \phi(t,\vec{x}) \to \phi(t,-\vec{x}) \qquad P : \mathcal{L} \to \mathcal{L}$$
$$P' : \phi(t,\vec{x}) \to -\phi(t,-\vec{x}) \qquad P : \mathcal{L} \to \mathcal{L}.$$

Either of these mappings provides a satisfactory definition of parity as they are related by $U_{P'} = (-1)^N U_P$. The action of the former on the creation and annihilation operators is $U_P^\dagger a_{\vec{k}} U_P = a_{-\vec{k}}$ and $U_P^\dagger a_{\vec{k}}^\dagger U_P = a_{-\vec{k}}^\dagger$. If we were to add an odd term such as $-h\phi^3$ to the Lagrangian, only P would remain a symmetry.

Example 2. Mixed Scalars and Pseudoscalars

The first term of the Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\sum_{a=1}^{4}\left((\partial_\mu\phi^a)^2 - \frac{1}{2}\mu_a^2(\phi^a)^2\right) - g\epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi^1\partial_\nu\phi^2\partial_\lambda\phi^3\partial_\sigma\phi^4,$$

is symmetric under both definitions of parity, P and P'. However, if an even number of the fields transform as P and P', then the interaction term is not invariant—it will change sign since it necessarily contains three space derivatives. The theory is invariant if an odd number of fields transform as P and P'.

Example 3. Parity Violation

If the Lagrangian in example two has a term $\sum_{a=1}^{4}(\phi^a)^3$ added to it, then no definition of parity gives a symmetry.

Example 4. A Case Where $P^2 \neq 1$

Our final example is designed to illustrate the fact that P^2 need not give back the original states. The Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\sum_{a=1}^{4}\left((\partial_\mu\phi^a)^2 - \mu^2(\phi^a)^2\right)\partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi$$
$$- h\sum_{a=1}^{4}(\phi^a)^3 - g\epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi^1\partial_\nu\phi^2\partial_\lambda\phi^3\partial_\sigma\phi^4(\psi^2 + \psi^{\dagger 2})$$

has a parity symmetry of the form: $\phi^a \to \phi^a$, $\psi \to \pm i\psi$ and $\psi^\dagger \to \mp\psi^\dagger$. For either choice of sign, $U_P \neq 1$; for example $U_P^{\dagger 2}\psi U_P^2 = -\psi$.

Time Reversal

Time reversal does not correspond to a unitary transformation. Rather, it is given by an anti-unitary operator Ω, that is an operator on the Hilbert space that satisfies for any a and b,

$$(\Omega a, \Omega b) = (b, a).$$

The reason that this produces a symmetry of the quantum theory is that a symmtry only needs to leave probabilities invariant, and not the inner products which are not measurable. Anti-unitary operators are also anti-linear:

$$\Omega(\alpha a + \beta b) = \alpha^* \Omega a + \beta^* \Omega b,$$

where α, β are complex numbers. For a free field theory, the action of simulataneous parity and time reversal transformation (PT) is $\Omega_{PT}^{-1}\phi(x)\Omega_{PT} = \phi(-x) = \phi(-t, -\vec{x})$. This transformation takes

$$\Omega_{PT}^{-1} a_{\vec{k}} \Omega_{PT} = a_{\vec{k}} \qquad \Omega_{PT}^{-1} a_{\vec{k}}^\dagger \Omega_{PT} = a_{\vec{k}}^\dagger \qquad \Omega_{PT}^{-1} i \Omega_{PT} = -i.$$

Scattering Theory

The basic problem in scattering theory is to find the amplitude that a state $|\psi\rangle$ in the far past becomes the state $\langle\psi'|$ in the far future. The overlap of these two states defines the scattering matrix S,

$$\langle\psi'|S|\psi\rangle \equiv {}^{\text{out}}\langle\psi'|\psi\rangle^{\text{in}}$$

which is an operator on the Hilbert space of states. One method of ensuring that the states do not interact in the distant past and distant future is to explicitly include a function $f(t)$ that becomes zero in these limits and then write the Hamiltonian as

$$H = H_0 + f(t)H',$$

where H_0 is the free Hamiltonian and H' contains the interactions. This formulation of scattering theory suggests that we must develop a time-dependent scattering theory.

The Schödinger Picture

In the Schrödinger picture, the operators are time-independent and the states evolve in time according to

$$|\psi(t)\rangle_S = U(t,t')|\psi(t')\rangle_S$$
$$i\frac{d}{dt}U(t,t') = H(p_S, q_S, t)U(t,t').$$

The Heisenberg Picture

In this description of the theory, the states do not evolve: $|\psi(t)\rangle_H = |\psi(0)\rangle_H = |\psi(0)\rangle_S$. However, the operators are time dependent:

$$A_H(t) = U(0,t)A_S(t)U^\dagger(0,t) = U^\dagger(t,0)A_S(t)U(t,0).$$

The Interaction Picture

Intermediate between these two pictures is the interaction picture for which the time evolution of the states is determined by the interaction part ($H'(t)$) of the Hamiltonian: $H(t) = H_0 + H'(t)$. The states are related to the Schrödinger picture states by $|\psi(t)\rangle_I = e^{iH_0(p_S,q_S)t}|\psi(t)\rangle_S$ and evolve as solutions to the differential equation,

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I.$$

Here $H_I(t)$ is the interaction part of the Hamiltonian with the operators in the interaction picture:

$$H_I(t) \equiv H'(p_I, q_I, t)$$
$$A_I(t) = e^{iH_0(p_S,q_S)t}A_S(t)e^{-iH_0(p_S,q_S)t}.$$

In field theory, $H_I(t)$ contains the free fields,

$$\phi(t,\vec{x}) = e^{iH_0 t}\phi(t,\vec{x})e^{-iH_0 t}.$$

We would like to define an interaction picture unitary operator that evolves the fields in time, $|\psi(t)\rangle = U_I(t,t')|\psi(t')\rangle_I$. This operator satisfies the differential equation,

$$i\frac{d}{dt}U_I(t,t') = H_I(t)U(t,t') \quad \text{with} \quad U_I(t,t')|_{t=t'} = 1.$$

The reason that $U_I(t,t')$ is so important is that it corresponds to the scattering operator introduced earlier, $S = U_I(\infty, -\infty)$. Thus, once we have solved for $U_I(t,t')$, we have solved scattering theory. The solution to the differential equation for this operator is given by **Dyson's Formula**:

$$U_I(t,t') = Te^{-i\int_{t'}^{t} dt''\, H_I(t'')} \qquad t > t'.$$

If the interactions are sufficiently weak, we can expand the exponential and approximate $U_I(t,t')$ by the first few terms. We shall show how scattering theory can be applied to three models.

<u>Model 1</u> $\qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\rho(x)\phi(x).$

Model 2 $\qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\rho(\vec{x}).$

Model 3 $\qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi - g\psi^\dagger\psi\phi.$

Wick's Theorem & Perturbation Theory — Fourth Week

Wick's Theorem

The *contraction* of two fields is the difference between their time-ordered product and their normal-ordered product:

$$\overbracket{A(x)B}(y) \equiv T(A(x)B(y)) - :A(x)B(y):.$$

For the spinless meson of mass μ this Wick contraction is equal to

$$\overbracket{\phi(x)\phi}(y) = \int \frac{d^4k}{(2\pi)^4}\, e^{\pm ik\cdot(x-y)} \frac{i}{k^2-\mu^2+i\epsilon}.$$

Wick's theorem provides a means of expanding the time-ordered product of fields in terms of a sum of field contractions and normal-ordered products:

$$\begin{aligned} T(\phi_1\cdots\phi_n) = {} & :\phi_1\cdots\phi_n: + :\overbracket{\phi_1\phi_2}\phi_3\cdots\phi_n: + \text{ terms with one contraction} \\ & + :\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\phi_5\cdots\phi_n: + \text{ terms with two contractions} \\ & + \cdots + \begin{cases} :\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\cdots\overbracket{\phi_{n-1}\phi_n}: & \text{if } n \text{ is even} \\ :\overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\cdots\overbracket{\phi_{n-2}\phi_{n-1}}\phi_n: & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

Diagrammatic Perturbation Theory

The solution to the S-matrix is given by Dyson's formula,

$$S = U_I(\infty,-\infty) = Te^{-i\int dt\, H_I(t)} = Te^{-i\int d^4x\, \mathcal{H}_I(t)}.$$

In model 3, the interaction Hamiltonian density is $gf(t)\psi^*\psi\phi$; so if g (which is called the coupling constant) is sufficiently small, we might hope to obtain a

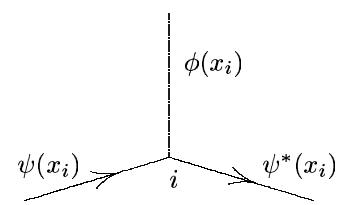
good approximation to the S-matrix by expanding the exponential and explicitly evaluating the first few terms. In all but a few theories which can be solved exactly, this perturbative approach the only method available to estimate S. Wick's theorem has the advantage of expanding the time ordered product in such a way to make clear the physical interpretation for each term, while at the same time eliminating processes that create and then eliminate mesons not present in the initial state. For example, one of the terms in Dyson's formula for model 3 is

$$-\frac{1}{2}g^2\int d^4x_1 d^4x_2\, f(t_1)f(t_2) :\psi^*(x_1)\phi(x_1)\phi(x_2)\psi(x_2): \overbracket{\psi^*(x_2)\psi}(x_1).$$

This term contributes to the following scattering processes:

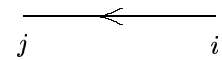
$$\begin{aligned} N+\phi &\to N+\phi, \\ \bar{N}+\phi &\to \bar{N}+\phi, \\ N+\bar{N} &\to \phi+\phi, \text{ and} \\ \phi+\phi &\to N+\bar{N}. \end{aligned}$$

Since the terms, first from expanding the exponential and then applying Wick's theorem to the time-ordered product of operators, quickly proliferate, we introduce a diagrammatic representation. These diagrams refer to integrals in position space and are called (in this course) Wick diagrams to differentiate them from the Feynman diagrams we shall introduce later. In model 3, for each factor $\psi^*(x_i)\psi(x_i)\phi(x_i)$ is associated a vertex,



while contractions correspond to connecting these vertices, *e.g.*

$\overbracket{\psi^*(x_i)\psi}(x_j) \;=$



Model 1 Solved

In model 1, for which the interaction Hamiltonian describes the emission of a meson from a source, $\mathcal{H}_I = g\rho(x)\phi(x)$, there are only two types of connected Wick diagrams:

$$\mathcal{O}_1 = \overset{1}{\bullet}\!\!\!-\!\!\!

The tildes denote Fourier transforms. Recall that as the characteristic width of a function inceases, its Fourier transform becomes more narrow. Thus as T, which describes the duration of the turning on and off function $f(t)$, increases, its Fourier transform $\tilde{f}(\omega_{\vec{k}})$ becomes a sharply peaked function centered about $\omega_{\vec{k}} = 0$. However, since physical mesons have $\omega_{\vec{k}} \geq \mu$, we must have $\mathcal{O}_1 = 0$ and thus

$$S = 1.$$

This equation just expresses the physical fact that a static source cannot impart any energy, in particular, to the mesons.

Feynman Diagrams — Fifth Week

Ground State Energy in Model 2

Although Model 2 does not contain any scattering, it still provides information about a system of two nucleons. Consider the example of a source with two nearly point-like sources at positions $\vec{x}_1$ and $\vec{x}_2$:

$$\rho(\vec{x}) = \text{“}\delta\text{”}(\vec{x} - \vec{x}_1) + \text{“}\delta\text{”}(\vec{x} - \vec{x}_2).$$

In the $T \to \infty$ limit, the ground state energy of this system, E_0, is equal to diagram (2); that is, the renormalization condition $\langle 0|S|0\rangle = 1$ for the vacuum actually tell us the energy of the ground state of a meson-source distribution:

$$\begin{aligned} E_0 &= -\frac{g^2}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\,|\tilde{\rho}(\vec{k})|^2 \frac{1}{\vec{k}^2 + \mu^2} \\ &= \frac{1}{2}\int d^3\vec{x} d^3\vec{y}\,\rho(\vec{x})\rho(\vec{y})V(\vec{x} - \vec{y}). \end{aligned}$$

The potential has been converted to a function in position space

$$V(\vec{x} - \vec{y}) = -g^2 \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 + \mu^2} = -\frac{g^2}{4\pi}\frac{e^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|}$$

and is known as the *Yukawa Potential*. The physical ground state ($|0\rangle_P$) wave function can be expressed in terms of the basis states of Fock space. The ground state is found by applying our solution to model 1 with

$$\rho(t, \vec{x}) = \begin{cases} \rho(\vec{x})e^{\epsilon t} & t < 0 \\ 0 & t > 0 \end{cases}.$$

As in model 1, we discover

$$\begin{aligned} \langle \vec{k}_1, \ldots, \vec{k}_n|0\rangle_P &= \langle \vec{k}_1, \ldots, \vec{k}_n|U_I(\infty, -\infty)|0\rangle \\ &= e^{-(\alpha - i\beta)/2} f(\vec{k}_1)\cdots f(\vec{k}_n) \end{aligned}$$

where (for $\epsilon \to 0$)

$$f(\vec{k}) = \frac{(-ig)\tilde{\rho}(\omega_{\vec{k}}, \vec{k})}{(2\pi)^{3/2}(2\omega_{\vec{k}})^{1/2}} \to \frac{-g\tilde{\rho}(\vec{k})}{(2\pi\omega_{\vec{k}})^{3/2}\sqrt{2}} \qquad \alpha = \int d^3\vec{k}\,|f(\vec{k})|^2.$$

When the “nucleons” (represented by the static meson-source) shrink to point-particles, $\alpha \to \infty$ logarithmically and the ground state no longer lies in the Fock space. Another sort of divergence occurs when the mass of the meson vanishes. Then $\langle N\rangle = \alpha \to \infty$; however, the energy and momentum in the meson field are both finite so that most of the mesons must have infinitesimally little energy.

Mass Renormalization

When a particle moves through some medium, it responds to external forces as though its mass is given by an effective mass which may be quite different from its mass in a vacuum. This effective mass is the sum of its mass in vacuo plus the effects of the medium on the particle as it moves through it. Some examples, beginning with one from the mid-nineteenth century, include

1. Hydrodynamics: the effective mass of a ball moving through a fluid equals its mass in vacuo plus one half of the mass of the fluid it displaces.
2. Lorentz's model for the electron: the mass of the electron is the sum is its rest mass plus the mass of its electric field.
3. Model 3:

The mass renormalization of model 3 is important for our discussion of scattering theory since should the masses change upon turning on the interactions, particles set up in the distant past to scatter at $t \approx 0$ may no longer do so. Therefore, so that the masses agree in both the free and interacting theories, we take μ and m to be the physical mass (which is the same at all times). To accomplish this feature, we write the Lagrangian for model 3 in the following form:

$$\begin{aligned} \mathcal{L} = &\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi \\ &+ f(t)\left[-g\psi^*\psi\phi + a + \tfrac{1}{2}b\phi^2 + c\psi^*\psi\right]. \end{aligned}$$

The vacuum energy density counterterm a, the meson mass counterterm b and the nucleon mass counterterm c are determined by the conditions that

$$\langle 0|S|0\rangle = 1 \Rightarrow \text{determines } a$$

and that single meson or single nucleon states do not scatter:

$$\langle \vec{k}|S|\vec{k}'\rangle = \delta^3(\vec{k}-\vec{k}') \Rightarrow \text{determines } b$$
$$\langle \vec{p}|S|\vec{p}'\rangle = \delta^3(\vec{p}-\vec{p}') \Rightarrow \text{determines } c.$$

The coefficients of the ϕ^2 and the $\psi^*\psi$ terms in the Lagrangian are sometimes called the *bare masses* of the meson and the nucleon:

$$\mu_0^2 \equiv \mu^2 - b \qquad m_0^2 \equiv m^2 - c.$$

The advantage of writing the counterterms with the interaction term is that then all of the quantities that we shall calculate in perturbation theory will already be expressed in terms of the physical masses.

Feynman Diagrams

Feynman diagrams offer a clever set of rules for writing down all the graphs in perturbation theory that contribute to a particular scattering process. Before introducing the Feynman rules for model 3, we shall study an example: consider the scattering of two nucleons. The scattering amplitude is determined by

$$\langle p_1', p_2'|(S-1)|p_1,p_2\rangle = \langle p_1',p_2'|T(e^{-ig\int d^4x\, \psi^*\psi\phi - \frac{b}{2g}\phi^2 - \frac{c}{g}\psi^*\psi}) - 1|p_1,p_2\rangle.$$

The states are normalized to transform correctly under Lorentz transformations:

$$|p\rangle = b^\dagger(p)|0\rangle \qquad b^\dagger(p) = (2\pi)^{3/2}(2\omega_{\vec{p}})^{1/2} b^\dagger_{\vec{k}}.$$

The first term in $S-1$ that contributes to nucleon-nucleon scattering is an order g^2 term:

$$-\frac{g^2}{2!}\int d^4x_1 d^4x_2 :\psi^*(x_1)\psi(x_1)\overbracket{\phi(x_1)\psi^*(x_1)\psi(x_2)}\phi(x_2):. \tag{1}$$

The reason that such a term contributes is that we must have two nucleon annihilation (creation) operators to absorb the initial (final) state $|p_1,p_2\rangle$ ($|p_1',p_2'\rangle$). Evaluating the operator (1) between these states gives

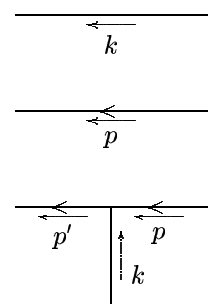
$$= -g^2\int d^4x_1 d^4x_2 \int \frac{d^4k}{(2\pi)^4}\Big[e^{ix_1\cdot(p_1'-p_1+k)}e^{ix_2\cdot(p_2'-p_2+k)} + e^{ix_1\cdot(p_2'-p_1+k)}e^{ix_2\cdot(p_1'-p_2+k)}\Big]\frac{i}{k^2-\mu^2+i\epsilon}.$$

Performing the integrations yields

$$\left[\frac{-g^2}{(p_1-p_1')^2-\mu^2+i\epsilon} + \frac{-g^2}{(p_1-p_2')^2-\mu^2+i\epsilon}\right] i(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2'). \tag{2}$$

We now introduce the Feynman rules for model 3 and show that we obtain the same answer for nucleon-nucleon scattering.

Feynman Rules:

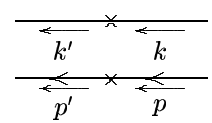


$$= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-\mu^2+i\epsilon}$$

$$= -ig\,(2\pi)^4\,\delta^4(p'-p-k)$$

Feynman Rules for the Counterterms:



$$= ia\,(2\pi)^4\,\delta^4(0)$$

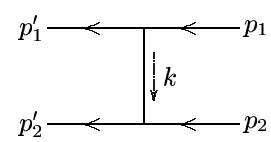
$$= ib\,(2\pi)^4\,\delta^4(k-k')$$

$$= ic\,(2\pi)^4\,\delta^4(p-p')$$

The prescription for applying these rules is to write down all of the diagrams to a given order that contribute to a process (determined by the external lines) and apply these rules to convert the graph into a function, $\mathcal{A}_{fi}$, of the external momenta. The contribution to the scattering matrix is then given by

$$\langle p_1',\ldots|(S-1)|p_1,\ldots\rangle = i\,\mathcal{A}_{fi}\,(2\pi)^4\delta^4(p_{f,\text{total}} - p_{i,\text{total}}).$$

For example, the two graphs that contribute to $N+N \to N+N$ at second order in g are



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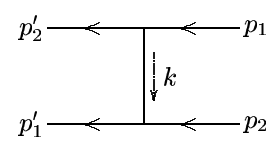


and applying the above rules immediately gives the same expression as in equation (2).

Let us try to interpret this result in the language of non-relativistic quantum mechanics. In the center of mass frame, the momenta are $p_{1,2} = (\sqrt{p^2+m^2}, \pm p\hat{e})$ and $p'_{1,2} = (\sqrt{p^2+m^2}, \pm p\hat{e}')$ and let us define $\cos\theta = \hat{e}\cdot\hat{e}'$. Then the Feynman amplitude is

$$\mathcal{A}_{fi} = g^2\left[\frac{1}{\Delta^2+\mu^2} + \frac{1}{\Delta_c^2+\mu^2}\right]$$

where

$$\Delta^2 = 2p^2(1-\cos\theta) \qquad \Delta_c^2 = 2p^2(1+\cos\theta).$$

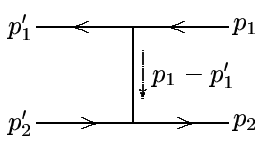
The first term represents a Yukawa potential while the second represents an exchange Yukawa potential.

Crossing Symmetry and Phase Space — Sixth Week

Survey of Scattering in Model 3

We complete our survey of all the scattering processes in Model 3 to order g^2. Since $N+N\to N+N$ was introduced last week, we begin with nucleon-antinucleon scatter for which there are two contributing graphs:

$N+\bar{N}\to N+\bar{N}$: Nucleon-Antinucleon Scattering



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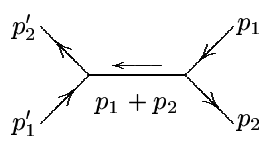


The Feynman rules for Model 3 produce an invariant amplitude,

$$i\mathcal{A} = (-ig)^2\left[\frac{i}{(p_1-p'_1)^2-\mu^2} + \frac{i}{(p_1+p_2)^2-\mu^2}\right],$$

in which the first term is just a Yukawa potential in the non-relativistic limit. The second term also arises in non-relativistic quantum mechanics. To see what it is, observe that $(p_1+p_2)^2 = E_{\text{tot}}^2$ where E_{tot} is the total center of mass energy. In the non-reltivistic limit then,

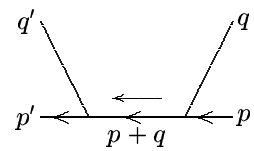
$$\frac{1}{(p_1+p_2)^2-\mu^2} \approx \frac{1}{2m-\mu}\frac{1}{E_{\text{tot}}-\mu}.$$

If we regard the virtual meson as a state with energy μ, then near threshold, such a term arises at second order in the Born approximation to the scattering matrix:

$$\mathcal{A} \propto \langle f|V|i\rangle + \sum_n \frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_{\text{tot}} - E_n \pm i\epsilon} + \cdots.$$

Consequently, we refer to the contribution from the second diagram as an energy eigenstate pole.

$N+\phi\to N+\phi$: Nucleon-Meson Scattering



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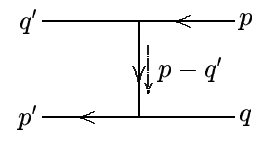
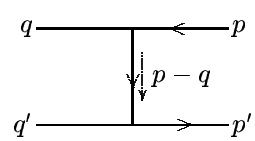


Here the invariant amplitude is

$$i\mathcal{A} = (-ig)^2\left[\frac{i}{(p+q)^2-\mu^2} + \frac{i}{(p-q')^2-\mu^2}\right],$$

an energy eigenstate pole and an exchange Yukawa potential, respectively. Note, however, that the range of the potential is not $1/m$ but rather $1/\sqrt{\mu(2m-\mu)}$ in the non-relativistic limit.

$N+\bar{N}\to\phi+\phi$: Nucleon-Antinucleon Annihilation



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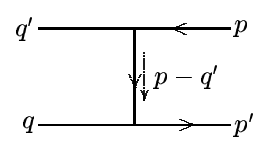


The invariant amplitude for this process,

$$i\mathcal{A} = (-ig)^2\left[\frac{i}{(p-q)^2-m^2+i\epsilon} + \frac{i}{(p-q)^2-m^2+i\epsilon}\right],$$

and these terms corespond respectively to a Yukawa potential and an exchange Yukawa potential.

Crossing Symmetry

Let us denote a generic two-paricle scattering process by the amplitude $\mathcal{A}(p_1,p_2,p_3,p_4)$ where all of the momenta are directed inward. Which phyical process this represents depends upon the 0th component of the momentum: if

$p_i^0 > 0$ then particle i is an initial particle, otherwise if $p_i^0 < 0$, then it is a final state. For example, in the scattering $1 + 2 \to \bar{3} + \bar{4}$ we have

$$p_1^0, p_2^0 > 0 \qquad \text{and} \qquad p_3^0, p_4^0 < 0.$$

Formally, we can always let $\mathcal{A}(p_1, p_2, p_3, p_4)$ describe the scattering processes $1 + 2 \to \bar{3} + \bar{4}$, $1 + 3 \to \bar{2} + \bar{4}$ and $1 + 4 \to \bar{2} + \bar{3}$ since the regions in momentum space corresponding to each scattering process are disjoint.

The set of sixteen variables (p_1, p_2, p_3, p_4) is overcomplete since we know that the scattering of two particles should require only two variables to describe—the center of mass energy and the scattering angle for example. A useful set of variables is provided by the Mandelstam variables,

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &\equiv (p_1 + p_3)^2 = (p_2 + p_4)^2 \\ u &\equiv (p_1 + p_4)^2 = (p_2 + p_3)^2, \end{aligned}$$

which represent the center of mass energy in each of the three channels:

$$\begin{aligned} 1 + 2 &\to \bar{3} + \bar{4} && s\text{-channel} \\ 1 + 3 &\to \bar{2} + \bar{4} && t\text{-channel} \\ 1 + 4 &\to \bar{2} + \bar{3} && u\text{-channel.} \end{aligned}$$

Of course, these variables are not all independent; they satisfy the constraint

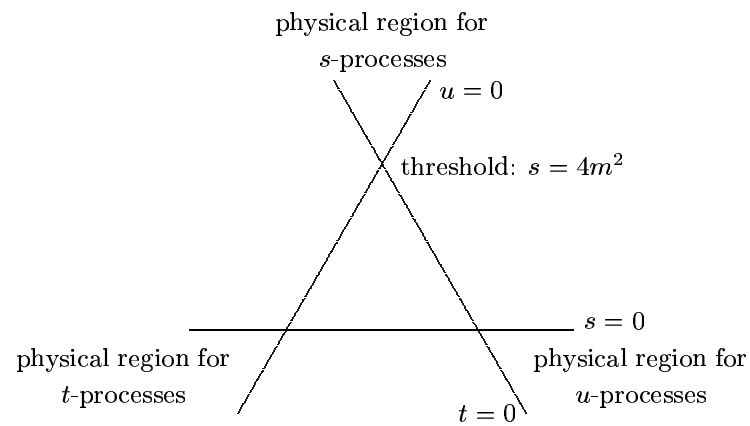
$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

We can introduce a two-dimensional plot for these variables by defining

$$\begin{aligned} s &= \vec{r} \cdot \hat{e}_s + \tfrac{1}{3} \sum_a m_a^2 \\ t &= \vec{r} \cdot \hat{e}_t + \tfrac{1}{3} \sum_a m_a^2 \\ u &= \vec{r} \cdot \hat{e}_u + \tfrac{1}{3} \sum_a m_a^2 \end{aligned}$$

where $\hat{e}_s$, $\hat{e}_t$ and $\hat{e}_u$ are unit vectors separated by 120°. In the case where all the masses of the external particles are equal, then the *Mandelstam-Kibble* plot

looks like



We now see that the three regions are not independent but must be connected by analytically continuing from one region to another. Although we can analytically continue to different regions, the physical meaning of the process changes:

s-region: energy eigenstate pole	$\to$	u-region: Yukawa potential	$\to$	t-region: exchange Yukawa

The TCP Theorem.

The TCP Theorem for our amplitude $\mathcal{A}(p_1, p_2, p_3, p_4)$ simply states that

$$\mathcal{A}(p_1, p_2, p_3, p_4) = \mathcal{A}(-p_1, -p_2, -p_3, -p_4).$$

More generally, any theory with a Lagrange density that is Lorentz invariant is CPT invariant, although it may violate C, P and T separately.

Relativistic Phase Space

When a theory is to be compared to an experiment, in addition to the invariant amplitude, $\mathcal{A}$, the rate at which a process occurs depends upon the available phase space. We here summarize the important formulae for physical decay and scattering processes.

1. Decays.

Let $d\Gamma$ denote the differential decay probability per unit time, then

$$d\Gamma = \frac{1}{2E} |\mathcal{A}|^2 \, (2\pi)^4 \delta^4(p_f - p_i) \prod_{\substack{\text{final} \\ \text{particles}}} \frac{d^3 \vec{k}_f}{(2\pi)^3 2E_f}.$$

The decay probability per unit time is then $m\Gamma/E$.

2. Cross Sections.

Let $d\sigma$ denote the differential transition probability per unit time per unit flux for two initial particle scattering; then

$$d\sigma = \frac{1}{4E_1E_2}|\mathcal{A}|^2 \frac{1}{|\vec{v}_1 - \vec{v}_2|}(2\pi)^4\delta^4(p_f - p_i) \prod_{\substack{\text{final}\\ \text{particles}}} \frac{d^3\vec{k}_f}{(2\pi)^3 2E_f}.$$

The total cross section is the integral over the available phase space:

$$\sigma = \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1 - \vec{v}_2|} \sum_{\substack{\text{final}\\ \text{states}}} \int \prod_{\substack{\text{final}\\ \text{particles}}} \frac{d^3\vec{k}_f}{(2\pi)^3 2E_f}(2\pi)^4\delta^4(p_f - p_i)\,|\mathcal{A}|^2.$$

3. A two body final state in the center of mass frame.

The phase space factor for the final states,

$$D \equiv (2\pi)^4\delta^4(p_f - p_i) \prod_{\substack{\text{final}\\ \text{particles}}} \frac{d^3\vec{k}_f}{(2\pi)^3 2E_f}. \tag{1}$$

becomes for a two particle final state

$$D = \frac{1}{16\pi^2}\frac{p_1\, d\Omega_1}{E_{\text{tot}}}$$

in the center of mass frame. Here 1 and 2 refer to the final particles. For example, the scattering of two particles into two particles has the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\text{tot}}^2}\frac{p_f}{p_i}|\mathcal{A}|^2,$$

in the center of mass frame.

The decay of a meson at rest into a $N\bar{N}$ pair in Model 3 proceeds with a decay width

$$\Gamma = \frac{g^2}{16\pi\mu^2}\sqrt{\mu^2 - 4m^2}.$$

4. The Optical Theorem.

The optical theorem in quantum field theory is just a consequence of $S^\dagger S = 1$. This theorem states that

$$2\,\text{Im}\,\mathcal{A}_{ii} = \sum_{\substack{\text{intermediate states}\\ \text{with } n_m \text{ particles}}} \frac{1}{n_m!}\int D_m|\mathcal{A}_{im}|^2$$

where D_m is the final state phase factor, introduced in equation (1). Physically, the optical theorem states that the decrease of a scattering beam in the forward direction corresponds to the probability that a particle scattered out of the beam.

5. Three body final state in the center of mass frame.

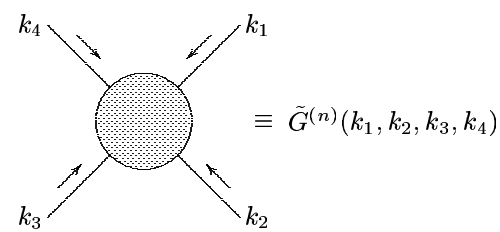
Finally, if we have three particles in the final state, the phase space factor is given by

$$D \equiv \frac{1}{256\pi^5}\frac{p_1\, dp_1}{E_1}, \frac{p_2\, dp_2}{E_2}\, d\Omega_1 d\phi_{12},$$

where 1 and 2 refer to two of the final state particles (the momentum of the third cancels these two in the center of mass frame) and ϕ_{12} is the azimuthal angle about $\vec{p}_1$.

Scattering Theory Reconsidered

We would now like to define scattering theory without recourse to the adiabatic function $f(t)$. The fundamental object that shall study in this reformulation of scattering theory is $\tilde{G}^{(n)}(k_1, \ldots, k_n)$ which represents all allowed graphs in a theory with n external lines. Graphically, we represent $\tilde{G}^{(4)}(k_1, \ldots, k_4)$ by



Since this object is of fundamental importance for our reformulation of scattering theory, we shall provide three interpretations for $\tilde{G}^{(n)}(k_1, \ldots, k_n)$.

First Interpretation.

The first place that such a graph could naturally be encountered is as an internal component of a larger Feynman diagram. One pleasant feature of this function $\tilde{G}^{(n)}(k_1, \ldots, k_n)$ is that if we explicitly remove the external lines by multiplying $\tilde{G}^{(n)}$ by an inverse propagator for each external leg, we should recover the scattering amplitude for a process with a total of n initial plus final particles. The example for two-particle scattering is

$$\langle k_1', k_2'|(S-1)|k_1, k_2\rangle = \prod_{a=1}^{4}[-i(k_a^2 - \mu^2)]\,\tilde{G}^{(4)}(k_1, k_2, -k_1', -k_2').$$

Since the k_a correspond to physical particles, these momenta should be on the mass shell: $k_a^2 = m^2$. This seemingly trivial result we shall encounter later and is called the *Reduction Formula*.

Second Interpretation: Green's Functions

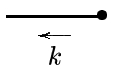
The function $\tilde{G}^{(n)}(k_1, \ldots, k_n)$ also has a role similar to that of the Green's functions introduced in electrodynamics. In the following, we shall write these functions in both momentum and position space with the following Fourier transform convention:

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \tilde{f}(k)$$
$$\tilde{f}(k) = \int d^4x\, e^{-ik\cdot x} f(x).$$

Let us introduce a real source into our theory,

$$\mathcal{L} \to \mathcal{L} + \rho(x)\phi(x);$$

this new interaction is accompanied by the following Feynman rule:



$$= i\tilde{\rho}(-k)$$

In the presence of an arbitrary source, the vacuum to vacuum amplitude is no longer uninteresting since the source could produce any number of mesons which interact and are later absorbed again by the source. We can thus formally expand the vacuum to vacuum scattering amplitude terms of the number of times the source produces a particle:

$$\begin{aligned}\langle 0|S|0\rangle\big|_\rho &= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\, \tilde{\rho}(-k_1)\cdots\tilde{\rho}(-k_n)\tilde{G}^{(n)}(k_1, \ldots, k_n) \\ &= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, \rho(x_1)\cdots\rho(x_n) G^{(n)}(x_1, \ldots, x_n)\end{aligned}$$

Notice that by taking the nth functional derivative of the vacuum to vacuum amplitude with respect to the source which is then set equal to zero, we extract the $G^{(n)}(x_1, \ldots, x_n)$. For this reason, we call

$$Z(\rho) \equiv \langle 0|S|0\rangle\big|_\rho$$

a *generating functional* for all the Green's functions. The generating functional for the connected Green's functions is

$$W(\rho) = \ln Z(\rho).$$

Real Scattering Theory

We are now ready to set $f(t) = 1$ and attempt to construct a scattering theory using the Green's functions and connecting this formulation to our previous version of scattering theory. Before doing so, we define the physical vacuum state, $|0\rangle_P$, which has zero energy and momentum and represents the only normalizable energy eigenstate in the theory. We add a source to the theory and define the Green's functions in terms of the generating functional as before:

$$Z(\rho) \equiv {}_P\langle 0|U(\infty, -\infty)|0\rangle_P\big|_\rho.$$

Then for the Green's functions, $G^{(n)}$, derived from this generating functional,

1. $\tilde{G}^{(n)}$ is given by a sum of Feynman graphs.
2. Up to a renormalization of the fields, the reduction formula is true.

In proving the first statement, we discover a third interpretation for $G^{(n)}$ in terms of a time-ordered product of Heisenberg fields:

$$G^{(n)}(x_1, \ldots, x_n) = {}_P\langle 0|T[\phi_H(x_1)\cdots\phi_H(x_n)]|0\rangle_P.$$

In proving that the reduction formula still holds without the adiabatic function, we shall introduce a renormalized field operator,

$$\phi'(x) \equiv Z_3^{-1/2}\left[\phi(x) - \langle 0|\phi(x)|0\rangle\right]$$

where

$$\sqrt{Z_3} = \langle k|\phi(0)|0\rangle.$$

This renormalized field has the following properties:

$$\langle 0|\phi'(x)|0\rangle = 0$$
$$\langle k|\phi'(x)|0\rangle = e^{ik\cdot x} \qquad \langle 0|\phi'(x)|k\rangle = e^{-ik\cdot x}.$$

Just as we saw that the Green's functions can be defined in terms of a vacuum expectation value of time-order Heisenberg fields, we can define renormalized Green's functions by

$$\tilde{G}^{(n)\prime} = \int d^4x_1 \cdots d^4x_n\, e^{-ik_1\cdots x_1} \cdots e^{-ik_1\cdots x_1} \langle 0|T[\phi'(x_1)\cdots\phi'(x_n)]|0\rangle.$$

It is this Green's function which satisfies the *Lehmann-Symanzik-Zimmermann (LSZ) reduction formula*:

$$\langle k_1', \ldots, k_m'|(S-1)|k_1, \ldots, k_n\rangle = \prod_{a=1}^{r}(-i)(k_a'^2 - \mu^2)\prod_{b=1}^{n}(-i)(k_b^2 - \mu^2)$$
$$G^{(n+m)\prime}(-k_1', \ldots, -k_m', k_1, \ldots, k_n).$$

To prove the LSZ reduction formula, we introduce wave packets given by

$$|f\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})}\,|k\rangle\langle k|f\rangle$$
$$f(x) = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})}\,e^{-ik\cdot x}\,\langle k|f\rangle;$$

we recover plane wave solutions by taking $|f\rangle \to |k\rangle$ ($f(x) \to e^{-ik\cdot x}$). These wave packets should satisfy the Klein-Gordon equation, $(\partial_\nu\partial^\nu - \mu^2)f(x) = 0$, and in terms of them the LSZ reduction formula looks like

$$\langle f_3, f_4|(S-1)|f_1, f_2\rangle$$
$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4\, f(x_1)f(x_2)f^*(x_3)f^*(x_4)(i)^4$$
$$(\partial_\nu\partial^\nu\big|_{x_1} - \mu^2)\cdots(\partial_\nu\partial^\nu\big|_{x_4} - \mu^2)\langle 0|T[\phi'(x_1)\phi'(x_2)\phi'(x_3)\phi'(x_4)]|0\rangle.$$

Let us introduce the following notation, for an operator A we define A^f by

$$A^f(t) \equiv i\int d^3\vec{x}\,\left(A\partial_0 f - f\partial_0 A\right).$$

This operator A^f naturally appears in the little lemma,

$$i\int d^4x\, f(x)(\partial_\nu\partial^\nu - \mu^2)A(x) = \left[\lim_{t\to\infty} - \lim_{t\to-\infty}\right] A^f(t).$$

When applied to the renormalized field, we have an object that creates or annihilates a single meson state:

$$\langle 0|\phi'^f(t)|0\rangle = 0$$
$$\langle k|\phi'^f(t)|0\rangle = \langle k|f\rangle$$
$$\langle 0|\phi'^f(t)|k\rangle = 0$$

For a multi-particle final state $|n\rangle$ ($P^\mu|n\rangle = p_n^\mu|n\rangle$),

$$\langle n|\phi'^f(t)|0\rangle = \langle n|\phi'(0)|0\rangle\frac{\omega_{\vec{k}} + p_n^0}{2\omega_{\vec{k}}}e^{i(p_n^0 - \omega_{\vec{k}})t}.$$

With the assistance of the Riemann-Lebesgue theorem, we then have for a final state $\langle\psi|$,

$$\lim_{t\to\pm\infty}\langle\psi|\phi'^f(t)|0\rangle = \langle\psi|f\rangle.$$

We can then prove the LSZ reduction formula above using

$$\lim_{t_1\to\infty}\lim_{t_2\to\infty}\langle\psi|\phi'^{f_1}(t_1)\phi'^{f_2}(t_2)|0\rangle = \langle\psi|f_1 f_2\rangle^{\text{out}}$$
$$\lim_{t_1\to-\infty}\lim_{t_2\to-\infty}\langle\psi|\phi'^{f_1}(t_1)\phi'^{f_2}(t_2)|0\rangle = \langle\psi|f_1 f_2\rangle^{\text{in}}.$$

For theorems of the production of soft photons, it is useful to use a "stop half-way" version of the LSZ formula:

$$\langle k_1', k_2'|(S-1)|k_1, k_2\rangle$$
$$= \int d^4x_1 d^4x_2\, e^{ik_1'\cdot x_2}e^{-ik_1\cdot x_1}(i)^2$$
$$(\partial_\nu\partial^\nu\big|_{x_1} + \mu^2)(\partial_\nu\partial^\nu\big|_{x_2} + \mu^2)\langle k_2'|T[\phi'(x_1)\phi'(x_2)]|k_2\rangle.$$

Renormalization of Model 3 — Seventh Week

Renormalization of Model 3

The general formulation of scattering theory in terms of the Green's functions can be applied to our Model 3. We could work directly with the 'bare' Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\psi^*\psi\phi,$$

but such an approach would not be very efficient since once we had evaluated some scattering process in terms of these bare parameters, we would still need to re-express them in terms of the physical parameters before we could compare the result to experiments. Moreover, we still need to renormalize ϕ and ψ since

these are the fields that appear in the Reduction Formula. Therefore, a better form for the Lagrangian is

$$\mathcal{L} =: \tfrac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \tfrac{1}{2}\mu^2\phi'^2 + \partial_\mu\psi'^*\partial^\mu\psi' - m^2\psi'^*\psi' - g\psi'^*\psi'\phi' : + \mathcal{L}_{\text{c.t.}},$$

where $\mathcal{L}_{\text{c.t.}}$ represents the set of counterterms:

$$\mathcal{L} =: A\phi' + \tfrac{1}{2}B\partial_\mu\phi'\partial^\mu\phi' - \tfrac{1}{2}C\phi'^2 + D\partial_\mu\psi'^*\partial^\mu\psi' - E\psi'^*\psi' - F\psi'^*\psi'\phi' + \text{constant}:$$

In expressing the Lagrangian thus, we have nine parameters instead of the original three of the theory, so we should augment the theory with six renormalization conditions:

1. $\langle 0\|\phi'\|0\rangle = 0$	4. The mass of the meson is μ
2. $\langle k\|\phi'(0)\|0\rangle = 1$	5. The mass of the nucleon is m .
3. $\langle 0\|\psi'(0)\|p\rangle = 1$	6. g is fixed by some convention

Of these conditions, only the first is in terms of the Green's functions. It implies that tadpoles do not contribute whether the line connected to the tadpole is internal or external. Otherwise we must overcome three difficulties if we are to use this Lagrangian as a basis of scattering theory:

1. The counterterms contain derivative interactions.
2. Conditions 2–5 are not explicitly conditions on Green's functions.
3. We must find the best way to define g.

Derivative Interactions.

The difficulty in treating derivative interactions is threefold: $\Pi^\mu \neq \partial^\mu\phi$, $\mathcal{H}' \neq -\mathcal{L}'$, and $T[\partial_\mu\phi\cdots] \neq \partial_\mu T[\phi\cdots]$. Each of these sicknesses however is the others' cure. Later in the course, we shall develop another approach to quantum field theory which will allow us to use derivative interactions. For now simply note that for some interactions, we obtain the right scattering matrix if we treat $\mathcal{H}' = -\mathcal{L}'$ and take all derivatives outside the time-ordering symbol.

We can show that for derivative interactions such as those in our Model 3 example, those coming from a rescaling of a field, give the correct rescaling of the Green's functions when the naïve Feynman rule is used, *i.e.*

$$\phi'(x) = Z_3^{-1/2}\phi(x) \quad \Rightarrow \quad G^{(n)\prime} = Z_3^{-n/2}G^{(n)}.$$

This result is true even in a more general theory,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \sum_{r=3}^{N} g_r\phi^r,$$

and the proof uses the conservation of ends:

$$n + 2I = \sum_r rV_r.$$

Expressing the Renormalization Conditions.

The renormalization conditions for the masses of the particles need to be interpreted in terms of the Green's functions, which are the only objects we know how to compute in perturbation theory. To begin let us insert into the Wightman Function† a complete set of intermediate states:

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \sum\int \langle 0|\phi'(x)|n\rangle\langle n|\phi'(y)|0\rangle.$$

From the first renormalization condition, $|n\rangle = |0\rangle$ does not contribute; only the single meson and multi-particle states contribute:

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \Delta_+(x-y;\mu^2) + \sum_{\substack{\text{all}\\ \text{else}}}\int \frac{d^3\vec{p}}{(2\pi)^3}(2\pi)^3\delta^4(p-p_n)\left|\langle 0|\phi'(0)|n\rangle\right|^2 e^{-ip_n\cdot(x-y)}.$$

If we introduce a function $\sigma(p^2)$ defined by

$$\sigma(p^2)\theta(p_0) \equiv \sum_{\substack{\text{all}\\ \text{else}}}(2\pi)^3\delta^4(p-p_n)\left|\langle 0|\phi'(0)|n\rangle\right|^2,$$

then we obtain the *Lehmann-Källen spectral representation* for the Wightman function:

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \Delta_+(x-y;\mu^2) + \int da^2\,\sigma(a^2)\Delta_+(x-y;a^2).$$

The function $\sigma(a^2)$ is always non-negative and is equal to zero up to some threshold above the one-meson state ($a^2 \leq (\mu+\delta)^2$). If we define the "renormalized propagator" by

$$\tilde{G}^{(n)\prime}(k,k') \equiv (2\pi)^4\delta^4(k+k')\,D'(k^2),$$

then

$$D'(k^2) = \frac{i}{k^2-\mu^2+i\epsilon} + \int da^2\,\sigma(a^2)\frac{i}{k^2-a^2+i\epsilon}.$$

† The Green's function contains a time-ordering operator.

The renormalization condition 4 (and analogously 5) can now be stated precisely,

1. $D'(k^2)$ has a pole at $k^2 = \mu^2$
2. with residue i.

In practice, there is even a more useful method for expressing these mass renormalization conditions involving 1PI (one particle irreducible) Green's functions. A 1PI Green's function is the sum over all 1PI graphs which do not include the propagators for the external legs. A graph is 1PI if

1. it contains at least one vertex,
2. is connected and
3. cannot be disconnected by cutting a single line.

As an example, consider the set of all 1PI graphs with two external legs:

$$\equiv (2\pi)^4\delta^4(k+k')\,[-i\Pi'(k^2)]$$

where $\Pi'(k^2)$ is called the *self-energy*. Then the propagator becomes, upon summing a geometric series of graphs,

$$D'(k^2) = \frac{i}{k^2 - \mu^2 - \Pi'(k^2) + i\epsilon},$$

so that the renormalization conditions become in terms of the self-energy

$$1.\ \ \Pi'(\mu^2) = 0 \qquad \text{and} \qquad 2.\ \ \left.\frac{d\Pi'(k^2)}{dk^2}\right|_{k^2=\mu^2} = 0.$$

Thus, if we express the counterterms order by order in perturbation theory,

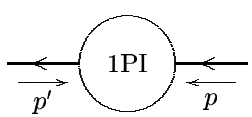
$$B = \sum B_n g^n \qquad C = \sum C_n g^n,$$

then these two conditions on $\Pi'(k^2)$ determine the two unknowns, B_n and C_n, that appear at order g^n in sum of 1PI graphs with two external lines.

The nucleon counterterms,

$$\mathcal{L}_{\text{c.t.}} = \cdots + D\partial_\mu\psi'^*\partial^\mu\psi' - E\psi'^*\psi' + \cdots,$$

can be similarly determined using the nucleon self-energy, $\Sigma'(p^2)$ which is defined by



$$\equiv (2\pi)^4\delta^4(p+p')\,[-i\Sigma'(p^2)]$$

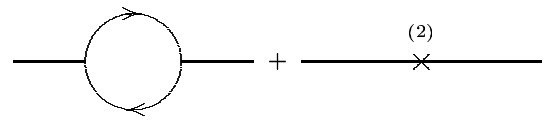
The conditions on the propagator—that it should have a pole at the physical mass m with residue i—translate into

$$1.\ \ \Sigma'(m^2) = 0 \qquad \text{and} \qquad 2.\ \ \left.\frac{d\Sigma'(p^2)}{dp^2}\right|_{p^2=m^2} = 0.$$

An Example: $\Pi'(k^2)$ to Order g^2 in Model 3.

As a first application of our general scattering theory, let us calculate the meson self-energy to order g^2 in Model 3. To this order only two graphs contribute, a graph with one-loop and a counterterm graph:

$-i\Pi'(k^2) \equiv$



which becomes, upon applying the renormalization conditions,

$$\begin{aligned}\Pi'(k^2) &= \Pi'_f(k^2) - B_2 g^2 k^2 + C_2 g^2 \\ &= \Pi'_f(k^2) - (k^2-\mu^2)\left.\frac{d\Pi'_f}{dk^2}\right|_{\mu^2} - \Pi'_f(\mu^2).\end{aligned}$$

$\Pi'_f(k^2)$ is the function obtained for the loop graph from the Feynman rules:

$$-i\Pi'_f(k^2) = (-ig)^2\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2+i\epsilon}\,\frac{i}{(q+k)^2-m^2+i\epsilon}.$$

This integral presents us with three problems:

1. It is not spherically symmetric.
2. It is not Euclidean.
3. It is not convergent.

The lack of convergence is actually not a problem since $\Pi'_f(k^2) - \Pi'_f(\mu^2)$ is not divergent.

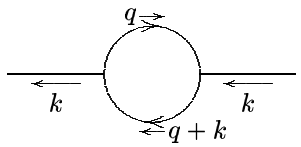
Evaluating Loop Integrals — Eight Week

The Self-Energy of the Meson in Model 3

The meson self-energy, $\Pi'(k^2)$, to lowest order is given by

$$\Pi'(k^2) = \Pi'_f(k^2) - \Pi'_f(\mu^2) - (k^2 - \mu^2)\frac{d\Pi'_f}{dk^2}\bigg|_{\mu^2}$$

where $\Pi'_f(k^2)$ represents the contribution from the one-loop diagram



With the momentum of the internal lines assigned as shown, the contribution of this one-loop graph is

$$-i\Pi'_f(k^2) = (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(q+k)^2 - m^2 + i\epsilon}.$$

We shall use this integral as an example of the general techniques used in evaluating the contributions of graphs involving loops to the Green's functions.

1. Feynman Parameters.

The first useful trick is to combine the two propagators into a single fraction. Feynman noticed that by introducing a new integration variable, the integrand could be put in the desired form:

$$\frac{1}{ab} = \int_0^1 dx \, \frac{1}{[ax + b(1-x)]^2}. \tag{1}$$

In the case of the meson self-energy, we have

$$\Pi'_f(k^2) = ig^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 + 2xq\cdot k + k^2x - m^2 + i\epsilon]^2}.$$

2. Shifting the Momentum.

The integrand is not yet spherically symmetric since it contains a term linear in q. However, this term can easily be eliminated by shifting the loop momentum by $q^\mu \to q'^\mu = q^\mu + xk^\mu$ so that $\Pi'_f(k^2)$ becomes

$$\Pi'_f(k^2) = ig^2 \int_0^1 dx \int \frac{d^4q'}{(2\pi)^4} \frac{1}{[q'^2 + k^2x(1-x) - m^2 + i\epsilon]^2}.$$

We shall drop the prime in the following calculations.

3. The Wick Rotation.

The integrand is now hyperbolically symmetric as the time component has the opposite signature as the spatial components. If there are no obstructions in the complex plane, we could rotate the time component of the loop momentum from along the real axis to along the imaginary axis: $q_0 = \pm iq_4$. Let us write the loop integral in the general form

$$I = \int \frac{d^4q}{(q_0^2 - \vec{q}^2 + a + i\epsilon)^n}.$$

The choice of the sign for the $i\epsilon$ shows that regardless of the sign of $\vec{q}^2 - a$, the only rotation that avoids the poles at $q_0 = \pm\sqrt{\vec{q}^2 - a - i\epsilon}$ is

$$q_0 \to iq_4.$$

This deformation of the contour of integration is called a *Wick Rotation*. After performing this rotation, we have a genuinely spherically symmetric integrand in terms of the Euclidean momentum q_E:

$$I = \int \frac{d^4q_E}{(-q_E^2 + a + i\epsilon)^n}.$$

4. Spherically Symmetric Integrals in Arbitrary Dimensions.

The momentum integral that appears in the expression for the self-energy of the meson is of the general form

$$I(d,r) = \int \frac{d^dq_E}{(q_E^2 + a)^r}$$

where we have written the momentum in d dimensions. Most frequently, we shall take $d \to 4$ but often, if the integral diverges, it is useful to evaluate the integral with d nearly, but not quite, equal to 4. This allows us to separate the pieces that diverge in the $d \to 4$ limit from those that remain finite. The divergent terms are then canceled by the divergent pieces coming from the counterterms so that the final expression for $\Pi'(k^2)$ is finite. The result for $I(d,r)$ is then

$$I(d,r) = \pi^{d/2}\frac{\Gamma(r - \frac{d}{2})}{\Gamma(r)} a^{\frac{d}{2}-r}.$$

Here, $\Gamma(z)$ is the gamma function which for $z = n$, an integer, is

$$\Gamma(n+1) = n! \qquad \Gamma(z+1) = z\,\Gamma(z).$$

The Γ-function has poles at $z = 0, -1, -2, \ldots$ and its behavior near these poles is

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \frac{\epsilon}{2}\left(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1)\right) + \mathcal{O}(\epsilon^2)\right]$$

where $\psi(z) \equiv d\ln\Gamma(z)/dz$. If $r-2 \leq 0$ then $I(d,r)$ contains a pole in the $d \to 4$ limit. Thus the pole in the Γ-function extracts an order $d/2 - r$ piece of the $a^{\frac{d}{2}-r}$ term that remains finite in the $d \to 4$ limit:

$$a^{\frac{d}{2}-r} = e^{-(r-d/2)\ln a} = 1 - \left(r - \frac{d}{2}\right)\ln a + \cdots.$$

5. $\Pi'(k^2)$ to order g^2, Concluded.

Using the above result for $I(d,r)$ for the meson self-energy, we discover that $\Pi'_f(k^2)$ does contain a pole at $d = 4$ but that this pole is precisely canceled by the pole in $\Pi'_f(\mu^2)$ which originates in the counterterms. The final finite expression for $\Pi'(k^2)$ is then

$$\Pi'(k^2) = \frac{g^2}{16\pi^2}\int_0^1 dx\left[\ln\frac{k^2x(1-x) - m^2 + i\epsilon}{\mu^2x(1-x) - m^2 + i\epsilon} - \frac{(k^2-\mu^2)x(1-x)}{\mu^2x(1-x) - m^2 + i\epsilon}\right].$$

Note that in accord with the Lehmann-Källen spectral theorem, the self-energy has a branch cut beginning at $k^2 = 4m^2$. Actually, the cut should begin at $\min(4\mu^2, 4m^2)$ but a two meson intermediate state does not appear in $\Pi'(k^2)$ until $\mathcal{O}(g^6)$ in perturbation theory.

More Loops.

The Feynman parameter trick can be used to combine an arbitrary number of propagators. Consider a graph which contains I internal lines and L loops. The generalization of equation (1) is

$$\prod_{r=1}^{I}\frac{1}{a_r} = (I-1)!\int_0^1 dx_1\cdots dx_I\,\delta(1-\textstyle\sum x_r)\frac{1}{(\sum a_r x_r)^I};$$

thus a generic multi-loop integral, upon introducing the Feynman parameters, assumes the form

$$I = (I-1)!\int_0^1 dx_1\cdots dx_I\,\delta(1-\textstyle\sum x_r)\displaystyle\int d^4k_1\cdots d^4k_L\,\frac{1}{D^I};$$

where

$$D \equiv \sum_{i,j=1}^{L} A_{ij}\,k_i\cdot k_j + \sum_{i=1}^{L} B_i\cdot k_i + C.$$

By appropriately shifting the momenta and rotating them until the analogue of A_{ij} in the new coordinates is the unit matrix, the multi-loop integral can be written in the following form,

$$I = (I-1)!\int_0^1 dx_1\cdots dx_I\,\delta(1-\textstyle\sum x_r)\frac{1}{[\det A]^2}\displaystyle\int d^4k_1\cdots d^4k_L\,\frac{1}{(\sum k_i^2 - C')^I}$$

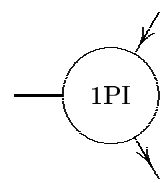
where

$$C' = C - \frac{1}{4}\sum_{i,j=1}^{L} B_i[A^{-1}]_{ij}B_j.$$

Once in this form, after Wick rotating $k_i^0 \to ik_i^4$, the multi-loop integral is a $4L$ dimensional spherically symmetric integral of the kind we encountered before.

The definition of g.

Only one of the six renormalization conditions for model 3 has not yet been expressed in terms of the Green's functions—the definition of the coupling constant g. To this end, we introduce another 1PI Green's function, $\Gamma(p^2, p'^2, q^2)$, defined as follows:



$$= -i(2\pi)^4\delta^4(q+p+p')\,\Gamma'(p^2, p'^2, q^2).$$

We then choose three fiducial values, $\underline{p}^2$, $\underline{p}'^2$ and $\underline{q}^2$, and define the renormalized coupling constant g by

$$g \equiv \Gamma(\underline{p}^2, \underline{p}'^2, \underline{q}^2).$$

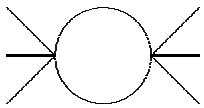
This condition fixes the F counterterms order by order in perturbation theory. For abstract proofs it is often convenient to choose $\underline{p}^2 = \underline{p}'^2 = \underline{q}^2 = 0$ but the most physical choice and the more useful in practice is to have all the particles on the mass hyperboloid: $\underline{p}^2 = \underline{p}'^2 = m^2$ and $\underline{q}^2 = \mu^2$.

Renormalization versus Infinities.

Interactions in a general quantum field theory fall into three classes:

1. If the coupling constant has positive mass dimension, it is a *super-renormalizable* interaction. An example is the $-g\psi^*\psi\phi$ interaction of model 3 in which g has the dimensions of mass.
2. Interactions with dimensionless coupling constants are *renormalizable*. An example is the interaction $-\frac{1}{4!}\lambda\phi^4$.
3. An interaction with negative mass dimension is said to be *non-renormalizable*.

The interaction $g\phi^5$ is non-renormalizable. If we were to include it in a theory, then we would find that the graph



is logarithmically divergent so that our theory would require a counterterm of the form $h\phi^6$. This interaction in turn would require a counterterm of the form $j\phi^7$ to cancel the infinities that it would create. To cure all the divergences of a theory with a non-renormalizable interaction requires an infinite number of counterterms and a theory with an infinite number of adjustable parameters has no predictive power.

Non-renormalizable interactions are allowed in an *effective field theory*. Such a theory is the low energy limit of a more fundamental theory and the non-renormalizable interactions that appear in the effective theory depend upon the finite number parameters of the higher energy theory. The reason such effective field theories are useful is that while they may contain an infinite number of interactions, these interactions only depend upon a small set of parameters and so the theory has predictive power. Moreover, interactions involving many fields are often suppressed by more powers of the massive particles of the high energy theory so that in practice, very few of the interactions of the effective theory are needed to study a given scattering or decay process.

Representations of $SO(3,1)$ — Ninth Week

Fields with Spin.

Until now, we have developed quantum field theory exclusively for spinless particles. However, the most fundamental particles observed in nature have spin $\frac{1}{2}$ or spin one so we must also develop a relativistic description of such particles. The existence of particles with spin is a manifestation of the underlying Lorentz

symmetry of spacetime. If we make a Lorentz transformation, $\Lambda \in SO(3,1)$, on a quantum field $\phi^a(x)$, the components of this field will in general mix:

$$U^\dagger(\Lambda)\phi^a(x)U(\Lambda) = D^a_b(\Lambda)\phi^b(\Lambda^{-1}x)$$
$$U^\dagger(\Lambda)\phi^a(0)U(\Lambda) = D^a_b(\Lambda)\phi^b(0);$$

here we restrict to linear transformations. If $\phi^a(x)$ were a tensor field, we would know the form of $D^a_b(\Lambda)$ but we would like to keep the notation general since there may exist fields that transform satisfactorily under a Lorentz transformations but are not tensor fields. Let us write the action of a Lorentz boost in a more succinct form:

$$U^\dagger(\Lambda)\phi(x)U(\Lambda) = D(\Lambda)\phi(\Lambda^{-1}x).$$

Applying two successive transformations and using that $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$, the matrices $D(\Lambda)$ satisfy,

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2).$$

We also impose $D(1) = 1$. A linear transformation from the elements of a group to a set of matrices that act on a d-dimensional vector space which satisfies these two properties is said to form a (d-dimensional) *representation* of the group. Therefore, stated more precisely, in order to find all the allowed fields with spin we must find all the representations of $SO(3,1)$.

Among the possible representations are some trivial ones. For example, we could always form a new representation $D'(\Lambda)$ from another one $D(\Lambda)$ by making a similarity transformation, $D'(\Lambda) = SD(\Lambda)S^{-1}$. We then say that D' is *equivalent* to D:

$$D \sim D'.$$

Another way to construct a representation from two representations, $D^{(1)}$ and $D^{(2)}$, is by forming the *direct sum* defined as follows:

$$[D^{(1)} \oplus D^{(2)}](\Lambda) = \begin{pmatrix} D^{(1)}(\Lambda) & 0 \\ 0 & D^{(2)}(\Lambda) \end{pmatrix}.$$

If the dimensions of the summands are respectively d_1 and d_2, then the direct sum representation is $d_1 + d_2$ dimensional. A representation that cannot be written as the direct sum of smaller representations is said to be *irreducible*. Therefore, our task is simpler—to find complete set of inequivalent, irreducible representations of $SO(3,1)$.

Representations of $SO(3)$.

As a first exercise in finding the irreducible representations of a group, we study the group of rotations, $SO(3)$. Not only does this group represent an important subgroup of the Lorentz group, but more importantly, since $SO(3,1)$ can be broken into two copies of $SO(3)$, we shall have immediately found the complete set of representations of the Lorentz group as well.

Let us denote a (counterclockwise) rotation about the direction $\vec{e}$ by an angle θ as

$$R(\vec{e}\theta) \qquad 0 \le \theta \le \pi.$$

Note that since $R(\vec{e}\pi) = R(-\vec{e}\pi)$, $SO(3)$ is a ball of radius π with antipodes identified. We can define a vector of matrices, $\vec{J}$, by performing an infinitesimal transformation:

$$\frac{d}{d\theta} D(R(\vec{e}\theta))\Big|_{\theta=0} \equiv -i\vec{e}\cdot\vec{J}.$$

If we perform two successive rotations about the same axis,

$$D\left(R(\vec{e}\theta')\right) D\left(R(\vec{e}\theta)\right) = D\left(R(\vec{e}(\theta'+\theta))\right),$$

and differentiate with respect to θ' we find the following differential equation,

$$-i\vec{e}\cdot\vec{J}D(R(\vec{e}\theta)) = \frac{d}{d\theta}D(R(\vec{e}\theta)),$$

whose solution is

$$D(R(\vec{e}\theta)) = e^{-i\vec{J}\cdot\vec{e}\,\theta}.$$

Note that if $D \sim D'$, then $\vec{J} \sim \vec{J}'$. Also if $D = D^{(1)} \oplus D^{(2)}$ is a direct sum representation, the $\vec{J}$'s similarly decompose, $\vec{J} = \vec{J}^{(1)} \oplus \vec{J}^{(2)}$.

We can further restrict the properties of $\vec{J}$ by examining successive rotations:

$$D(R')^{-1}D(R(\vec{e}\theta))D(R') = D(R(R'^{-1}\vec{e}\theta)).$$

Differentiating with respect to θ gives

$$e^{i\vec{e}\cdot\vec{J}\theta}\vec{J}e^{-i\vec{e}\cdot\vec{J}\theta} = R(\vec{e}\theta)\vec{J}.$$

For small θ, identifying the $\mathcal{O}(\theta)$ terms of both sides we have

$$i[\vec{e}\cdot\vec{J},\vec{J}] = \vec{e}\times\vec{J}$$

which is more conventionally written as

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

We have found the commutation relations of the angular momentum.

The matrices $\vec{J}$ are the generators of the *Lie algebra* of $SO(3)$. To find the elements of the representation of the (Lie) group $SO(3)$, we can just exponentiate these matrices, $D(R(\vec{e}\theta)) = e^{-i\vec{e}\cdot\vec{J}\theta}$, except that in doing so D may be multiple valued. If the group is simply connected, that is every closed loop can be smoothly deformed to a point without obstruction, then exponentiation of the algebra gives the group. If we remember that $SO(3)$ is a ball with antipodes identified, then we see that a rotation by 2π represents a path which returns to the identity—a closed loop. However, such a path cannot be deformed to a point and so $SO(3)$ is not simply connected. If we continue and rotate by 4π we do obtain a path which can be shrunk to a point and so at worst we may have double valued representations.† We shall mention this property below when we enumerate the representations of $SO(3)$.

We know the complete list of inequivalent irreducible representations of the rotations from our study of quantum mechanics. These representations are labeled by their spin, s:

$$\vec{J}^{(s)} \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

where

$$J_z^{(s)}\,|m\rangle = m\,|m\rangle \qquad m = -s, -s+1, \ldots, s-1, s$$
$$D^{(s)} = e^{-i\vec{J}^{(s)}\cdot\vec{e}\,\theta}.$$

We recall some of the properties of $D^{(s)}$:

1. It is single-valued for $s = 0, 1, 2, \ldots$; double-valued for $s = \frac{1}{2}, \frac{3}{2}, \ldots$.
2. $\dim D^{(s)} = 2s+1$.
3. In the standard representation the $\vec{J}(s)$'s are Hermitian so that the $D^{(s)}$'s are unitary.
4. There is a similarity transformation that links a representation with its complex conjugate: $D^{(s)} \sim D^{(s)*}$ and $\vec{J}^{(s)} \sim -\vec{J}^{(s)*}$.
5. Vector Addition: If we have representations of dimension d_1 and d_2, we can from the *direct product* representation as follows:

$$\langle\alpha',\beta'|\,A\otimes B\,|\alpha,\beta\rangle = \langle\alpha'|\,A\,|\alpha\rangle\langle\beta'|\,B\,|\beta\rangle$$

which is of dimension d_1d_2. The direct product representation is not, in general, irreducible; for $SO(3)$ it decomposes into the following irreducible representations:

$$D^{(s_1)}(R)\otimes D^{(s_2)}(R) \sim D^{(s_1+s_2)}(R)\oplus D^{(s_1+s_2-1)}(R)\oplus\cdots D^{|s_1-s_2|}(R)$$
$$\sim \bigoplus\sum_{s=|s_1-s_2|}^{s_1+s_2} D^{(s)}(R)$$

† What we have basically described is the fact that $\pi_1(SO(3)) = \mathbf{Z}_2$.

which is the usual rule for the addition of angular momenta in quantum mechanics.

Representations of $SO(3,1)$.

In addition to the rotations, the Lorentz group contains boosts which we shall denote as $A(\vec{e}\phi)$ which is a boost in the direction $\vec{e}$ with rapidity ϕ ($0 \leq \phi < \infty$). Just as in the case of $SO(3)$, applying two rotations about the same directions, or two boosts in the same direction, allows us to derive the following differential equations:

$$i\frac{d}{d\theta}D(R(\vec{e}\theta))\Big|_{\theta=0} = \vec{e}\cdot\vec{L} \qquad i\frac{d}{d\phi}D(A(\vec{e}\phi))\Big|_{\phi=0} = \vec{e}\cdot\vec{M}$$

whose solutions are

$$D(R(\vec{e}\theta)) = e^{-i\vec{e}\cdot\vec{L}\theta} \qquad D(A(\vec{e}\phi)) = e^{-i\vec{e}\cdot\vec{M}\phi}.$$

The commutation relations for the generators L_i and M_i are

$$\begin{array}{ll} [L_i, L_j] = i\epsilon_{ijk}L_k & [L_i, M_j] = i\epsilon_{ijk}M_k \\ [M_i, L_j] = i\epsilon_{ijk}M_k & [M_i, M_j] = -i\epsilon_{ijk}L_k \end{array}.$$

This pattern suggests that if we define $\vec{J}^{\pm}$ by

$$\vec{J}^{\pm} = \tfrac{1}{2}[\vec{L} \pm i\vec{M}]$$

we shall find the following commutation relations:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm} \qquad [J_i^{\pm}, J_j^{\mp}] = 0.$$

Our representation of the Lie algebra of $SO(3,1)$ thus breaks into two representations of $SO(3)$. Therefore, from our study of $SO(3)$ above, we immediately know the representations of $SO(3,1)$:

$$D^{(s_+,s_-)}(\Lambda) \qquad s_{\pm} = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$
$$J_z^{\pm}\,|m_+, m_-\rangle = m_{\pm}\,|m_+, m_-\rangle.$$

We can write the generators of the representation in the direct produce notation:

$$\vec{J}^{+} = \vec{J}^{(s_+)} \otimes 1$$
$$\vec{J}^{-} = 1 \otimes \vec{J}^{(s_-)};$$

in this notation a rotation or a boost becomes

$$D(R(\vec{e}\theta)) = e^{-i[\vec{J}^{+}+\vec{J}^{-}]\cdot\vec{e}\,\theta}$$
$$D(A(\vec{e}\phi)) = e^{-[\vec{J}^{+}-\vec{J}^{-}]\cdot\vec{e}\,\phi}.$$

Observe that while the former is a unitary matrix, the latter is not.

We now list some of the properties of $D^{(s_+,s_-)}(\Lambda)$:

1. $\dim D^{(s_+,s_-)} = (2s_+ + 1)(2s_- + 1)$.
2. Rotations are Unitary: $D(s_+, s_-)^{-1}(R) = D(s_+, s_-)^{\dagger}(R)$.
   Boosts are Hermitian: $D(s_+, s_-)(A) = D(s_+, s_-)^{\dagger}(A)$.
3. $D^{(s_+,s_-)}$ is single (double) valued if $s_+ + s_-$ is even (odd).
4. $\vec{J}^{(s)} \sim -\vec{J}^{(s)*}$ and $D^{(s_+,s_-)*}(\Lambda) \sim D^{(s_-,s_+)}(\Lambda)$.
5. Under a parity transformation, $D^{(s_+,s_-)}(\Lambda) \to D^{(s_-,s_+)}(\Lambda)$.
6. The direct product of two representations decomposes as follows:

$$D^{(s_+^1,s_-^1)}(\Lambda) \otimes D^{(s_+^2,s_-^2)}(\Lambda) \sim \bigoplus \sum_{s_+=|s_+^1-s_+^2|}^{s_+^1+s_+^2} \sum_{s_-=|s_-^1-s_-^2|}^{s_-^1+s_-^2} D^{(s_+,s_-)}(\Lambda).$$

7. If $s_{\pm}^1 = s_{\pm}^2$ then under exchange, $D^{(s_+,s_-)} = D^{(2s_+^1,2s_-^1)}$ is symmetric, $D^{(2s_+^1-1,2s_-^1)}$ is antisymmetric, $D^{(2s_+^1,2s_-^1-1)}$ is antisymmetric, $D^{(2s_+^1-1,2s_-^1-1)}$ is symmetric, *etc.*
8. If we restrict to the $SO(3)$ rotation subgroup of $SO(3,1)$ a representation $D^{(s_+,s_-)}(R)$ has the following decomposition into irreducible representations of $SO(3)$:

$$D^{(s_+,s_-)}(R) \sim \bigoplus \sum_{s=|s_+-s_-|}^{s_++s_-} D^{(s)}(R).$$

In terms of this notation, the four dimensional vector representation of $SO(3,1)$ corresponds to the representation $(\frac{1}{2}, \frac{1}{2})$. When we restrict to the rotations, the vector decomposes into a one and a three dimensional representation of $SO(3)$—the time and the position vector, $(t, \vec{x})$.

Spinors and the Dirac Equation — Tenth Week

Spinors.

Let us call the two spinors that tranform as the $D^{(0,\frac{1}{2})}$ and $D^{(\frac{1}{2},0)}$ representations of the Lorentz group u_+ and u_- respectively:

$$\Lambda : u_+(x) \to D^{(0,\frac{1}{2})} u_+(\Lambda^{-1}x)$$
$$\Lambda : u_-(x) \to D^{(\frac{1}{2},0)} u_-(\Lambda^{-1}x).$$

Under rotations or boosts, $u_\pm$ transform explicitly as

$$R(\vec{e}\theta) : u_\pm(x) \to e^{-i\vec{\sigma}\cdot\vec{e}\theta/2} u_\pm(R^{-1}x)$$
$$A(\vec{e}\phi) : u_\pm(x) \to e^{\pm\vec{\sigma}\cdot\vec{e}\phi/2} u_\pm(A^{-1}x).$$

We would like to construct a Lagrangian from these fields which is Lorentz invariant; as a first step, we note that the following combination of u_+ and $u_+^\dagger$ transforms as a vector,

$$V^\mu = (u_+^\dagger u_+, u_+^\dagger \vec{\sigma} u_+).$$

Similarly,

$$V^\mu = (u_-^\dagger u_-, -u_-^\dagger \vec{\sigma} u_-).$$

also transforms as a vector. With these vectors we can now write down the Lagrangian, $\mathcal{L}(u_+, u_+^\dagger, \partial_\mu u_+, \partial_\mu u_+^\dagger)$, involving solely the u_+ spinor that satisfies the following conditions:

1. $\mathcal{L}$ is bilinear in its arguments
2. $\mathcal{L}$ is Lorentz invariant
3. $S = \int d^4x\, \mathcal{L}$ should be real
4. $\mathcal{L}$ contains no more than two derivatives
5. $\mathcal{L}$ is invariant under $u_+ \to e^{-i\lambda} u_+$.

We shall relax this last condition later in the course. The only Lagrangian consistent with all of these conditions is

$$\mathcal{L} = \pm i u_+^\dagger (\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) u_+.$$

For the u_- spinor, the Lorentz invariant Lagrangian is analogously:

$$\mathcal{L} = \pm i u_-^\dagger (\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) u_-.$$

The equation of motion following from the Lagrangian for the u_+ field is the Weyl equation:

$$(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) u_+ = 0;$$

notice that the solutions to this equation must also satisfy the Klein-Gordon equation:

$$(\partial_0^2 - \vec{\nabla}^2) u_+ = 0$$

and therefore a general solution to the Weyl equation can be written as a linear combination of plane wave solutions. As an example, for a particle moving along the z-axis ($k^\mu = (1,0,0,1)$) with a positive frequency,

$$u_+(x) = u_{\vec{k}} e^{-ik\cdot x}$$

is of the form

$$u_{\vec{k}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

while for u_- we would have

$$u_{\vec{k}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These particles move at the speed of light and so do not have a rest frame. The spin of these particles cannot be defined; nevertheless it is still possible to define the angular momentum about the direction of motion. The eigenvalue λ of this operator,

$$\Lambda \equiv \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|},$$

is called the *helicity* of the particle. If we assume that when we canonically quantize the theory we have $\langle 0|u_+(x)|k\rangle = u_{\vec{k}} e^{-ik\cdot x}$, then we conclude that u_+ has helicity $\lambda = \frac{1}{2}$. The antiparticles associated with u_+ have helicity $-\frac{1}{2}$. Actually this property is more general: if a theory is to be consistent with TCP then a particle must have the opposite helicity as its antiparticle.

The Dirac Equation.

With the u_+ field alone, we were unable to construct a Lagrangian for a massive spin $\frac{1}{2}$ particle that met our five requirements. Although we could contruct a massive Lagrangian for a Weyl spinor were we to remove the fifth condition, let us build a Lagrangian with both u_+ and u_- instead. The conditions that we impose on $\mathcal{L}(u_+, u_-, \partial_\mu u_+, \partial_\mu u_-, \text{conjugates})$ are as follows:

1. $\mathcal{L}$ is bilinear in its arguments
2. $\mathcal{L}$ is Lorentz invariant
3. $S = \int d^4x\, \mathcal{L}$ should be real
4. $\mathcal{L}$ is invariant under $u_\pm \to e^{-i\lambda} u_\pm$.
5. $\mathcal{L}$ is invariant under the following parity transformation:

$$P : u_+(t,\vec{x}) \to a\, u_-(t,-\vec{x})$$
$$u_-(t,\vec{x}) \to b\, u_+(t,-\vec{x})$$

6. $\mathcal{L}$ contains no more than one derivative.

The only Lorentz invariant terms with no derivatives are $u_+^\dagger u_-$ and its conjugate $u_-^\dagger u_+$. The terms with one derivative we constructed so that we can immediately write down the form of the Lagrangian:

$$\mathcal{L} = \pm\left[iu_+^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ + iu_-^\dagger(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})u_- - mu_+^\dagger u_- - mu_-^\dagger u_+\right].$$

Note that we have chosen the phase so that $m \geq 0$ is real. We have further imposed parity invariance which holds as long as $a(=b)$ is a complex number of modulus one. Using our $U(1)$ symmetry, we can choose $a = b = 1$ to give a unique definition of parity:

$$P : u_\pm(t, \vec{x}) \to u_\mp(t, -\vec{x}).$$

The equation of motion, the *Dirac equation*, for the pair u_+ and u_- is

$$\begin{aligned} i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ &= mu_- \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})u_- &= mu_+ \end{aligned}$$

and Weyl spinors thus both satisfy the massive Klein-Gordon equation:

$$(\partial_0^2 - \vec{\nabla}^2)u_\pm = -m^2 u_\pm.$$

Due to the complexity of the Dirac equation some notation has been developed to simplify the study of relativistic spinors. To begin, the two Weyl spinors can be combined into a single 4-component *Dirac bispinor*:

$$\psi \equiv \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

If we also define the accompanying 4×4 *Dirac matrices*, $\vec{\alpha}$ and β:

$$\vec{\alpha} = \vec{\alpha}^\dagger = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \qquad \beta = \beta^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the massive spinor Lagrangian and the Dirac equation become respectively

$$\mathcal{L} = \pm\left\{i\psi^\dagger(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi - m\psi^\dagger\beta\psi\right\}$$

and

$$i(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi = m\beta\psi.$$

One disadvantage of this representation, called the *Weyl representation* is that the mass term is not diagonal, so the representation which is more commonly used, and therefore is called the *Standard representation*, is

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} u_+ + u_- \\ u_+ - u_- \end{pmatrix}$$

with

$$\vec{\alpha} = \vec{\alpha}^\dagger = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \beta^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the Standard representation, the mass term is diagonal since it is multiplied by β.

Solutions to the Dirac Equation.

Positive Frequency Solutions: We now look for solutions of the form $\psi = u_{\vec{p}}e^{-ip\cdot x}$ where $p^0 = \sqrt{\vec{p}^2 + m^2}$ since the components of the spinors also satisfy the Klein-Gordon equation. In the rest frame ($\vec{p} = \vec{0}$) the Dirac equation becomes $u_{\vec{0}} = \beta u_{\vec{0}}$ which has two solutions:

$$u_{\vec{0}}^{(1)} = \sqrt{2m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_{\vec{0}}^{(2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

These solutions correspond respectively to particles with spin $\frac{1}{2}$ and spin $-\frac{1}{2}$ in the z-direction. If we boost these solutions in the z-direction, $\vec{p} = |\vec{p}|\hat{z}$, then we have

$$u_{\vec{p}}^{(1)} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} \qquad u_{\vec{p}}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}.$$

Negative Frequency Solutions: When a solution of the form $\psi = v_{\vec{p}}e^{ip\cdot x}$ is tried in the Dirac equation, the rest frame solutions are

$$v_{\vec{0}}^{(1)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_{\vec{0}}^{(2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

These solutions correspond respectively to anti-particles with spin $-\frac{1}{2}$ and spin $\frac{1}{2}$ in the z-direction.

For massive particles, for arbitrary $\vec{p}$, the normalization conditions for these positive and negative solutions of the Dirac equation are respectively,

$$u^{(r)\dagger}_{\vec{p}}\beta u^{(s)}_{\vec{p}} = \bar{u}^{(r)}_{\vec{p}} u^{(s)}_{\vec{p}} = 2m\,\delta^{rs}$$
$$v^{(r)\dagger}_{\vec{p}}\beta v^{(s)}_{\vec{p}} = \bar{v}^{(r)}_{\vec{p}} v^{(s)}_{\vec{p}} = -2m\,\delta^{rs}.$$

However, for massless particles, these normalization conditions become singular so it is more convenient to write them as

$$(u^{(r)\dagger}_{\vec{p}} u^{(s)}_{\vec{p}}, u^{(r)\dagger}_{\vec{p}}\vec{\alpha} u^{(s)}_{\vec{p}}) = 2p^\mu\,\delta^{rs}$$
$$(v^{(r)\dagger}_{\vec{p}} v^{(s)}_{\vec{p}}, v^{(r)\dagger}_{\vec{p}}\vec{\alpha} v^{(s)}_{\vec{p}}) = 2p^\mu\,\delta^{rs}.$$

The Dirac Algebra.

Although we have simplified the appearance of the Dirac equation with the introduction of the Dirac matrices we can yet simplify it further. Let us choose a representation in which the Dirac matrices are hermitian. The mass term $\psi^\dagger\beta\psi$ can be written in a more compact form in terms of the *Dirac adjoint*,

$$\bar{\psi} \equiv \psi^\dagger\beta,$$

first introduced by Pauli. The Dirac adjoint for a matrix A is

$$\bar{A} = \beta A^\dagger \beta = \gamma^0 A^\dagger \gamma^0.$$

If we write the vector costructed from the spinors in terms of the Dirac adjoint, we have

$$V^\mu = (\psi^\dagger\psi, \psi^\dagger\vec{\alpha}\psi) = (\bar{\psi}\beta\psi, \bar{\psi}\beta\vec{\alpha}\psi) = \bar{\psi}\gamma^\mu\psi$$

which suggests that we define the matrices $\gamma^0 = \beta$ and $\vec{\gamma}^i = \beta\vec{\alpha}$ which transforms as a vector,

$$\bar{D}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu.$$

These γ-matrices satisfy a simple algebra,

$$\{\gamma^\nu, \gamma^\nu\} = 2g^{\mu\nu},$$

which is a particular example of a Clifford algebra. The γ-matrices are not self-adjoint, $\gamma^{\mu\dagger} = \gamma_\mu$, but instead are Dirac self-adjoint:

$$\bar{\gamma}^\mu = \gamma^\mu.$$

Finally, we introduce a last bit of notation, introduced by Feynman,

$$\not{a} \equiv a_\mu\gamma^\mu = a^\mu\gamma_\mu.$$

The anti-commutation relation above for the γ-matrices means that

$$\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b \qquad \not{a}\not{a} = a^2.$$

With this new notation, the Dirac equation can finally be written in the following succinct form:

$$\mathcal{L} = \pm\bar{\psi}(i\not{\partial} - m)\psi$$

and our positive and negative solutions to the Dirac equation obey

$$\psi = u_{\vec{p}}e^{-ip\cdot x} \qquad (\not{p} - m)u_{\vec{p}} = 0$$
$$\psi = v_{\vec{p}}e^{ip\cdot x} \qquad (\not{p} + m)v_{\vec{p}} = 0.$$

Feynman Diagrams for Fermions — Eleventh Week

The Feynman Rules for a Theory with Fermions.

Let us generalize Model 3 by replacing the complex scalar fields with spin $\frac{1}{2}$ nucleons,

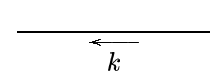
$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + i\bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\Gamma\psi\phi,$$

where Γ could be 1, γ_5 or some more general Dirac matrix. As before, the Feynman amplitude is defined by

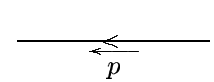
$$\langle f|(S-1)|i\rangle = i\mathcal{A}\,(2\pi)^4\delta^4(p_f - p_i)$$

and is computed using the following Feynman rules:

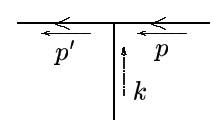
(1) Factors.



$$= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - \mu^2 + i\epsilon}$$



$$= \int \frac{d^4p}{(2\pi)^4}\frac{i}{\not{p} - m + i\epsilon}$$



$$= -i\Gamma\,(2\pi)^4\,\delta^4(p' - p - k)$$

$$\text{For every nucleon } \left\{ \begin{matrix} \text{in} \\ \text{out} \end{matrix} \right\}, \text{ a } \left\{ \begin{matrix} u \\ \bar{u} \end{matrix} \right\}$$

$$\text{For every antinucleon } \left\{ \begin{matrix} \text{in} \\ \text{out} \end{matrix} \right\}, \text{ a } \left\{ \begin{matrix} \bar{v} \\ v \end{matrix} \right\}$$

(2) Combining Factors.

a. Along a fermion line we start with the term corresponding to the head of the arrow on the left and multiply the additional terms we find as we follow the line back through the diagram.

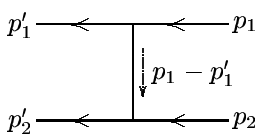
b. For a closed loop, we may start anywhere on the loop and include the Dirac matrices we find as we follow the line backwards (head to tail) around the loop. We take the trace of all the Dirac matrices for each loop.

c. **Include a factor of (-1) for each closed loop.**

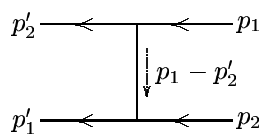
d. Use a convention consistently for the definition of initial and final states to determine the relative sign for graphs that contribute to the same process. For example, in $2\phi \to N\bar{N}$ we can define the final state as either $|f\rangle \propto b_2^\dagger c_1^\dagger |0\rangle$ or as $b_2^\dagger c_1^\dagger |0\rangle$ but once we have made a choice, we must use it consistently throughout the calculation of a Feynman amplitude.

$N + N \to N + N$: Nucleon-Nucleon Scattering.

Let us apply the above rules to find the invariant amplitudes for several typical scattering processes. To be more specific, the theory we shall study will be that of a pseudoscalar meson, $\mathcal{L}' = -g\bar{\psi} i\gamma_5 \psi\phi$, and the first process we examine is nucleon-nucleon scattering. To lowest order in g, the following two graphs contribute:



and



The amplitude for this process is then

$$i\mathcal{A} = (-ig)^2 \left\{ \bar{u}_{\vec{p}_1'} i\gamma_5 u_{\vec{p}_1} \, \bar{u}_{\vec{p}_2'} i\gamma_5 u_{\vec{p}_2} \frac{i}{(p_1 - p_1')^2 - \mu^2} \right.$$
$$\left. - \bar{u}_{\vec{p}_1'} i\gamma_5 u_{\vec{p}_2} \, \bar{u}_{\vec{p}_2'} i\gamma_5 u_{\vec{p}_1} \frac{i}{(p_1 - p_2')^2 - \mu^2} \right\}$$

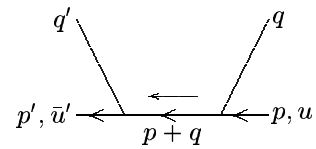
where the sign is fixed by choosing the initial and final states to be

$$|i\rangle \propto b_{\vec{p}_1}^\dagger b_{\vec{p}_2}^\dagger |0\rangle$$
$$|f\rangle \propto b_{\vec{p}_1'}^\dagger b_{\vec{p}_2'}^\dagger |0\rangle.$$

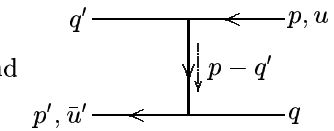
and is consistent with Fermi-Dirac statistics.

$N + \phi \to N + \phi$: Nucleon-Meson Scattering.

The two graphs which contribute to order g^2 for nucleon-meson scattering are



and



which produce the invariant amplitude

$$i\mathcal{A} = (-ig)^2 \bar{u}' \left\{ i\gamma_5 \frac{i(\not{p} + \not{q} + m)}{(p+q)^2 - m^2} i\gamma_5 + i\gamma_5 \frac{i(\not{p}' - \not{q} + m)}{(p'-q)^2 - m^2} i\gamma_5 \right\} u.$$

When we wish to calculate a differential cross section from an invariant amplitude, it is useful to simplify the amplitude as much as is possible before squaring it. In this amplitude, we can not only eliminate the γ_5 factors but some of the momenta as well since the spinors satisfy $(\not{p} - m)u = 0$ and $\bar{u}(\not{p}' - m) = 0$ so that the amplitude becomes

$$\mathcal{A} = g^2 \bar{u}' \not{q} u \left\{ \frac{1}{u - m^2} - \frac{1}{s - m^2} \right\}.$$

Spin Sums and Averages.

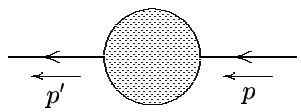
In a real experiment, often the actual spins of the products of an interaction are not measured and further, the initial particles are typically not prepared in a specific spin state. Therefore, in calculating the differential cross section we should sum over the final spins and average over the initial spins. In our last example process, we found that $\mathcal{A} = F(s,u)\, \bar{u}' \not{q} u$; summing and averaging over the spins gives

$$|\mathcal{A}|^2 = \frac{1}{2} |F(s,u)|^2 \sum_{r,s=1}^{2} \mathrm{Tr}\, [\bar{u}'^r \not{q} u^s \, \bar{u}^s \not{q} u'^r]$$
$$= \frac{1}{2} |F(s,u)|^2 \sum_{r,s=1}^{2} \mathrm{Tr}\, [\not{q} u^s \, \bar{u}^s \not{q} u'^r \bar{u}'^r]$$
$$= \frac{1}{2} |F(s,u)|^2 \, \mathrm{Tr}\, [\not{q}(\not{p} + m)\not{q}(\not{p}' + m)].$$

We have used the completeness relations in the final step.

The Spectral Representation of the Fermion Propagator.

The derivation of the spectral representation for the fermion propagator proceeds almost exactly as that for the meson propagator presented earlier in the course. The Fermion propagator corresponds to the following sum of graphs:



$$\equiv\ (2\pi)^4\delta^4(p-p')\,S'(p)$$

The spectral representation for the propagator is

$$S'(p) = \frac{i}{\not p - m + i\epsilon} + \int da^2\,\sigma_+(a^2)\frac{i(\not p + a)}{p^2 - a^2 + i\epsilon} + \int da^2\,\sigma_-(a^2)\frac{i(\not p - a)}{p^2 - a^2 + i\epsilon}$$

where $\sigma_\pm(a^2) \geq 0$.

The Renormalization Conditions.

The bare Lagrangian for the theory with a pseudoscalar meson,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu_0^2\phi^2 + \bar\psi(i\not\partial - m_0)\psi - g_0\bar\psi i\gamma_5\psi\phi,$$

is not expressed in terms of the masses and the coupling constant that an experimentalist would measure. Thus it is useful, as in model 3, to write the Lagrangian, and the perturbative expansion of the S-matrix, in terms of the physical masses and coupling constant (m, μ and g):

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi')^2 - \tfrac{1}{2}\mu^2\phi'^2 + \bar\psi'(i\not\partial - m)\psi' - g\bar\psi' i\gamma_5\psi'\phi' + \mathcal{L}_{\text{c.t.}},$$

where the counterterms are

$$\mathcal{L}_{\text{c.t.}} = \tfrac{1}{2}A(\partial_\mu\phi')^2 - \tfrac{1}{2}B\phi'^2 + \bar\psi'(iC\not\partial - D)\psi' - E\bar\psi' i\gamma_5\psi'\phi'.$$

The renormalized fields are related to those of the bare Lagrangian by

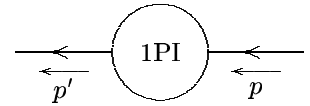
$$\phi'(x) = Z_3^{-1/2}\phi(x) \qquad \psi'(x) = Z_2^{-1/2}\psi(x).$$

As before in model 3, we require a set of renormalization conditions to fix the counterterms and to define the rescaled fields. The conditions on the meson propagator are the same as in model three so we concentrate on the fermion propagator. For a parity invariant theory, we impose the following condition on the fermion field:

$$\langle 0|\psi(0)|\vec 0, s_z = \tfrac{1}{2}\rangle = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the renormalization constant Z_2 ($\psi' = Z_2^{-1/2}\psi$) is $a = Z_2^{1/2}$ where by a phase choice on $|\vec 0, s_z = \frac{1}{2}\rangle$, $Z_2 > 0$.

The A and B counterterms are fixed as before by conditions on the meson self-energy so we shall write only the conditions for C and D. We define the fermion self-energy by the sum of one-particle irreducible graphs of the following form:



$$\equiv\ -i\Sigma'(\not p)$$

By summing the geometric series of insertions of the self-energy on a fermion line, we find that the propagator is

$$S'(\not p) = \frac{i}{\not p - m - \Sigma'(\not p)}.$$

Near $\not p = m$, the propagator is

$$S'(\not p) = \frac{i}{\not p - m + i\epsilon} + \text{analytic near } \not p = m$$

which requires that the self-energy satisfy

$$\Sigma'(m) = 0 \qquad \left.\frac{d\Sigma'(\not p)}{d\not p}\right|_{\not p = m} = 0;$$

these two conditions fix the C and D counterterms order by order in perturbation theory.

The Fermion Self-Energy.

As an example, to $\mathcal{O}(g^2)$, the fermion self-energy is

$$\begin{aligned}\Sigma'(\not p) &= \Sigma_f(\not p) + C_2\not p - D_2 \\ &= \Sigma_f(\not p) - \Sigma_f(m) - (\not p - m)\left.\frac{d\Sigma_f}{d\not p}\right|_{\not p = m}\end{aligned}$$

upon imposing the renormalization conditions on the self-energy. The integral $\Sigma_f(\not p)$ is the one-loop Feynman graph

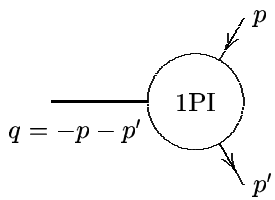
$$\Sigma_f(\not p) = ig^2\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\,\frac{\not p(1-x) - m}{[k^2 + p^2x(1-x) - m^2x - \mu^2(1-x) + i\epsilon]^2}.$$

Note that both the C_2 and the D_2 counterterms are required to cancel the divergences. In model 3, only the mass conterterm was need to cancel the

divergence in the self-energy. Here the situation has become more precarious. For the meson self-energy in this pseudoscalar theory with fermions, both the A and B terms are also necessary to cancel the divergences in the self-energy graph at $\mathcal{O}(g^2)$.

The Coupling Constant Renormalization.

The final renormalization condition is that which fixes the coupling constant counterterm E. We define the Green's function



$$= -i\Gamma'(p^2, p'^2, q^2).$$

upon which we shall impose the conditions that will define the coupling constant g. The first possible condition is to take $p = p' = q = 0$:

$$\Gamma'(0,0,0) = i\gamma_5 g.$$

A second condition is to project onto the mass shell to define g:

$$(\not{p}' + m)\Gamma'(m^2, m^2, \mu^2)(\not{p} + m) = ig(\not{p}' + m)\gamma_5(\not{p} + m).$$

The advantage of the latter is that if we consider nucleon-meson scattering,

$$\bar{u}_{\vec{p}'}\,\Gamma'(p', p+q)S'(p+q)\Gamma'(p+q, p)u_{\vec{p}},$$

by inserting

$$u_{\vec{p}} = \frac{\not{p} + m}{2m}u_{\vec{p}} \qquad \bar{u}_{\vec{p}'} = \bar{u}_{\vec{p}'}\frac{\not{p}' + m}{2m}$$

and using the second definition for g, we obtain the correct residue of the pole for a physical scattering process.

At one-loop, the logarithmic divergence of the leading Feynman graph is exactly canceled by the E counterterm.

C, P, T, and the CPT Theorem — Thwelfth Week

Parity.

The simplest discrete symmetry to extend to a field theory with fermions is parity. We built this symmetry into our fields when we derived the Dirac equation:

$$\begin{aligned} P : \psi(t,\vec{x}) &\to \beta\psi(t,-\vec{x}) \\ U_P^\dagger\psi(t,\vec{x})U_P &= \beta\psi(t,-\vec{x}). \end{aligned} \tag{1}$$

From this action on the field, we would like to determine the action of the unitary operator, U_P, on the creation and annihilation operators; to do so, we first note that

$$\beta u_{\vec{p}}^{(r)} = u_{-\vec{p}}^{(r)} \qquad \beta v_{\vec{p}}^{(r)} = -v_{-\vec{p}}^{(r)}$$

which alters the direction of the momentum but leaves the spin unchanged. Thus the right side of equation (1) is

$$\beta\psi(t,-\vec{x}) = \sum_{r=1}^{2}\int \frac{d^3\vec{p}}{(2\pi)^{3/2}(2E_{\vec{p}})}\left[b_{-\vec{p}}^{(r)}u_{\vec{p}}^{(r)}e^{-ip\cdot x} - c_{-\vec{p}}^{(r)\dagger}v_{\vec{p}}^{(r)}e^{ip\cdot x}\right]$$

where we have changed variables, $\vec{p} \to -\vec{p}$. Comparing this to

$$U_P^\dagger\psi(t,-\vec{x})U_P = \sum_{r=1}^{2}\int \frac{d^3\vec{p}}{(2\pi)^{3/2}(2E_{\vec{p}})}\left[U_P^\dagger b_{\vec{p}}^{(r)}U_P\,u_{\vec{p}}^{(r)}e^{-ip\cdot x} + U_P^\dagger c_{\vec{p}}^{(r)\dagger}U_P\,v_{\vec{p}}^{(r)}e^{ip\cdot x}\right]$$

provides the transformation properties of the creation and annihilation operators under parity:

$$U_P^\dagger\left\{\begin{array}{c} b_{\vec{p}}^{(r)} \\ b_{\vec{p}}^{(r)\dagger} \\ c_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)\dagger} \end{array}\right\}U_P = \left\{\begin{array}{c} b_{-\vec{p}}^{(r)} \\ b_{-\vec{p}}^{(r)\dagger} \\ -c_{-\vec{p}}^{(r)} \\ -c_{-\vec{p}}^{(r)\dagger} \end{array}\right\}.$$

The product of the parity of a fermion particle and its antiparticle is always -1, a fact which leads to observables consequences. In the annihilation process $N + N \to 2\pi$ (at rest), the $J = 1$ channel is allowed but the $J = 0$ channel is forbidden by parity conservation.

Charge Conjugation.

Recall that for a complex scalar field, $\phi(x)$, the action of charge conjugation was $\phi(x) \to \phi^*(x)$. If $\phi(x)$ was a solution of the Klein-Gordon equation then so too was its charge conjugate, $\phi^*(x)$. However, a similar claim can not be automatically made for the Dirac field $\psi(x)$,

$$(i\not{\partial} - m)\psi(x) = 0 \quad \not\Rightarrow \quad (i\not{\partial} - m)\psi^*(x) = 0$$

unless we can find a representation in which the γ-matrices are all purely imaginary:

$$\gamma^{\mu *} = -\gamma^{\mu}.$$

Such a representation is called a **Majorana representation** and an example of such a representation is provided by the choice,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \qquad \gamma^1 = i \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$$
$$\gamma^2 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^3 = i \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}.$$

In another representation, related to the above representation by a matrix S, the charge conjugate field is given by $\psi^c = \mathcal{C}\psi^*$, where $\mathcal{C} = SS^{*-1}$. In the Majorana representation, both M_i and L_i are purely imaginary so $D(\Lambda) = D^*(\Lambda)$. Stated differently, charge conjugation commutes with Lorentz transformations.

In order to see how the Dirac bispinors transform, we can choose the basis of the $v_{\vec{p}}^{(r)}$'s so that

$$u_{\vec{p}}^{(r)\,*} = v_{\vec{p}}^{(r)}.$$

Then equating the two sides of the equation $U_C^\dagger \psi(x) U_C = \psi^*(x)$,

$$U_C^\dagger \psi(x) U_C = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^{3/2}(2E_{\vec{p}})} \left[U_C^\dagger b_{\vec{p}}^{(r)} U_C \, u_{\vec{p}}^{(r)} e^{-ip\cdot x} + U_C^\dagger c_{\vec{p}}^{(r)\dagger} U_C \, v_{\vec{p}}^{(r)} e^{ip\cdot x} \right]$$
$$\psi^*(x) = \sum_{r=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^{3/2}(2E_{\vec{p}})} \left[b_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} + c_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} \right]$$

we find the transformation properties of the creation and annihilation operators:

$$U_C^\dagger \left\{ \begin{matrix} b_{\vec{p}}^{(r)} \\ b_{\vec{p}}^{(r)\dagger} \\ c_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)\dagger} \end{matrix} \right\} U_C = \left\{ \begin{matrix} c_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)\dagger} \\ b_{\vec{p}}^{(r)} \\ b_{\vec{p}}^{(r)\dagger} \end{matrix} \right\}.$$

Let us next examine the behavior of a nucleon-antinucleon state under charge conjugation. For comparison, in a scalar field theory, if

$$|\psi\rangle = \int d^3\vec{p}_1 d^3\vec{p}_2 \, f(\vec{p}_1, \vec{p}_2) b_{\vec{p}_1}^\dagger c_{\vec{p}_2}^\dagger |0\rangle,$$

then $U_C |\psi\rangle = \pm|\psi\rangle$ for $f(\vec{p}_1, \vec{p}_2) = \pm f(\vec{p}_2, \vec{p}_1)$. However, for fermions, since the creation operators anticommute, under charge conjugation of

$$|\psi\rangle = \sum_{r,s=1}^{2} \int d^3\vec{p}_1 d^3\vec{p}_2 \, f_{rs}(\vec{p}_1, \vec{p}_2) b_{\vec{p}_1}^{(r)\dagger} c_{\vec{p}_2}^{(s)\dagger} |0\rangle,$$

the sign is the opposite:

$$U_C |\psi\rangle = \pm|\psi\rangle \quad \text{for} \quad f_{rs}(\vec{p}_1, \vec{p}_2) = \mp f_{sr}(\vec{p}_2, \vec{p}_1).$$

This symmetry also leads to observable phenomena. The decay of para-positronium ($S = 0$) into two photons is allowed but the decay of ortho-positronium ($S = 1$) into two photons is forbidden by charge conjugation.

Fermion Bilinears and Charge Conjugation.

A general fermion bilinear may be written as $\bar{A}MB$, where M is a matrix and A, B are Dirac fields. Under charge conjugation,

$$U_C^\dagger : \bar{A}MB : U_C =: \bar{B}\bar{M}^* A:$$

and since

$$\bar{1}^* = 1 \qquad \bar{i}\gamma_5{}^* = i\gamma_5 \qquad \bar{\gamma}^{\mu *} = -\gamma^\mu \qquad \bar{\gamma}^\mu \gamma_5{}^* = \gamma^\mu \gamma_5 \qquad \sigma^{\bar{\mu}\nu *} = -\sigma^{\mu\nu},$$

we conclude that the fermion bilinears are odd or even under charge conjugation as follows:

$$\begin{array}{ll} \text{Even:} & \bar{\psi}\psi, \bar{\psi} i\gamma_5 \psi, \bar{\psi}\gamma^\mu \gamma_5 \psi \\ \text{Odd:} & \bar{\psi}\gamma^\mu \psi, \bar{\psi}\sigma^{\mu\nu}\psi. \end{array}$$

Note that a CP transformation satisfies

$$U_P U_C = U_C U_P U(R(\vec{e}\, 2\pi))$$

since, for fermions, particles and antiparticles always have the opposite parity.

Time Reversal.

As with scalar particles, it is simpler to work with a simultaneous PT transformation rather than with T alone. We shall also still work in the Majorana representation. Note that

$$PT : \psi(x) \to M\psi(-x)$$

will be a solution to the equations of motion $(i\partial\!\!\!/ - m)M\psi(-x) = 0$, provided $\{M, \gamma^\mu\} = 0$. This condition determines M up to a phase to be $M = i\gamma_5$. Let us construct the anti-unitary operator Ω_{PT} such that

$$\Omega_{PT}^{-1}\psi(x)\Omega_{PT} = i\gamma_5\psi(-x).$$

Regardless of the phase we have chosen, $\Omega_{PT}^2 = U(R(\vec{e}2\pi))$. Since under PT,

$$PT : \vec{p} \to \vec{p} \qquad \vec{s} \to -\vec{s}$$

we expect that the action of PT on the annihilation operators is

$$\Omega_{PT}^{-1}\left\{\begin{matrix} b_{\vec{p}}^{(1)} \\ b_{\vec{p}}^{(2)} \end{matrix}\right\}\Omega_{PT} = \left\{\begin{matrix} b_{\vec{p}}^{(2)} \times \text{some phase} \\ b_{\vec{p}}^{(1)} \times \text{some other phase} \end{matrix}\right\}.$$

In a Majorana basis, we have that

$$\begin{aligned} u_{\vec{p}}^{(r)\prime} &= -i\gamma_5 u_{\vec{p}}^{(r)*} \\ v_{\vec{p}}^{(r)\prime} &= -i\gamma_5 v_{\vec{p}}^{(r)*} \end{aligned}$$

so we can define the PT transformed annihilation operators by

$$\begin{aligned} \sum_{r=1}^{2} b_{\vec{p}}^{(r)}\, u_{\vec{p}}^{(r)} &= \sum_{r=1}^{2} b_{\vec{p}}^{(r)\prime}\, u_{\vec{p}}^{(r)\prime} \\ \sum_{r=1}^{2} c_{\vec{p}}^{(r)}\, v_{\vec{p}}^{(r)} &= \sum_{r=1}^{2} c_{\vec{p}}^{(r)\prime}\, v_{\vec{p}}^{(r)\prime}. \end{aligned}$$

Then for

$$\Omega_{PT}^{-1}\left\{\begin{matrix} b_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)} \end{matrix}\right\}\Omega_{PT} = \left\{\begin{matrix} b_{\vec{p}}^{(r)\prime} \\ c_{\vec{p}}^{(r)\prime} \end{matrix}\right\}$$

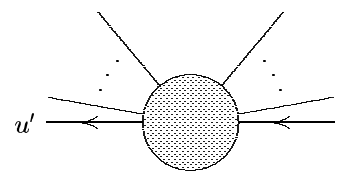
the field transforms correctly, $\Omega_{PT}^{-1}\psi(x)\Omega_{PT} = i\gamma_5\psi(-x)$.

The CPT Theorem.

The statement of the invariance of the S-matrix under CPT for a Klein-Gordon theory was equivalent to

$$\mathcal{A}(p_1, p_2, \ldots, p_n) = \mathcal{A}(-p_1, -p_2, \ldots, -p_n).$$

We shall now extend the CPT theorem to processes with one initial and one final Dirac particle with an arbitrary number of initial or final mesons. The invariant amplitude for such a process looks like



$$u \to \mathcal{A} = \bar{u}' M(p_1, p_2, \cdots, p_n) u$$

Under CPT the Dirac spinors transform as

$$u \overset{PT}{\to} i\gamma_5 u^* \overset{C}{\to} -i\gamma_5 u \qquad \bar{u}' \overset{CPT}{\to} \bar{u}'(-i\gamma_5)$$

so that under CPT the amplitude becomes

$$(-1)(-i)^2 \bar{u}'\gamma_5 M(-p_1, -p_2, \ldots, -p_n)\gamma_5 u;$$

the extra initial minus sign is a result of reordering the fermions for the CPT reversed process. Thus, showing CPT invariance is equivalent to demonstrating that

$$\bar{u}' M(p_1, p_2, \ldots, p_n) u = \bar{u}'\gamma_5 M(-p_1, -p_2, \ldots, -p_n)\gamma_5 u.$$

We shall prove this fact by appealing to Lorentz invariance and analyticity.

The amplitude is Lorentz invariant, so we can write

$$\bar{u}' M(p_1, p_2, \ldots, p_n) u = \bar{u}' \overline{D(\Lambda)} M(\Lambda p_1, \Lambda p_2, \ldots, \Lambda p_n) D(\Lambda) u.$$

The Lorentz transformation that we shall perform is

$$\Lambda = R(e_z\, \pi) A(e_z\, i\pi)$$

which involves an imaginary boost. We can make an imaginary boost provided everything is analytic. First note that $D(\Lambda)$ and $\overline{D(\Lambda)} = D(\Lambda)^{-1}$ are both analytic since for a boost

$$D(A(\vec{e}\phi)) = e^{\vec{\alpha}\cdot\vec{\sigma}\,\phi/2}$$

which is an analytic function of ϕ. If we write the integral M as

$$M(p_1, \ldots, p_n) = \int d^4k_1 \cdots d^4k_L \, \frac{N(p_1, \ldots; k_1, \ldots)}{D(p_1, \ldots; k_1, \ldots)},$$

the denominator $D(p_1, \ldots; k_1, \ldots)$ is Lorentz invariant while the numerator $N(p_1, \ldots; k_1, \ldots)$ is just a polynomial of the momenta which is analytic. Therefore everything is analytic so we are permitted to perform an imaginary boost $\Lambda = R(e_z\, \pi) A(e_z\, i\pi)$:

$$D(\Lambda) = \gamma_5 = D(\Lambda)^{-1} \qquad \Lambda p = -p.$$

Thus we see that the CPT theorem holds.