Concepts of Integrability in IOTA

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Integrable Particle Dynamics in Accelerators
January 2019 USPAS at Knoxville

(This talk owes much to S. Valishev’s HB2018 Talk & G. Stancari’s 2018 DOE GARD Review Talk)
Most particle accelerators are based around proximity to linear integrable optics. The linear accelerator Hamiltonian:

\[ H = H_x + H_y \]

\[ H_x = \frac{1}{2} p_x^2 + \frac{1}{2} K_x(s)x^2 \quad H_y = \frac{1}{2} p_y^2 + \frac{1}{2} K_y(s)y^2 \]

Each transverse degree of freedom is a linear oscillation with a time-dependent focusing. We can normalize the phase-space coordinates and write in terms of action-angle:

\[ x_N(s) = \frac{x(s)}{\sqrt{\beta_x(s)}} \quad p_{xN}(s) = \sqrt{\beta_x(s)}p_x(s) - \frac{\beta_x'(s)}{2\sqrt{\beta_x(s)}}x(s) \]

\[ \psi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')} \quad J_x(s) = \frac{1}{2\pi} \oint p_{xN}dx_N \]

No resonances, two degrees of freedom, two action invariants.
Aberrations on Linear Accelerator

Particle accelerators aren’t just perfect continuous focusing:
- Discrete elements lead to time-dependence (s-dependence).
- Sextupoles for chromaticity correction.
- Octupoles for nonlinear focusing.
- Magnet field quality & fringe-field effects.

There will be many higher-order resonances, tune-shift with amplitude, and x-y coupling.

\[ H = H_x + H_y + V(x, y, s) \rightarrow \nu_x J_x + \nu_y J_y + V(J_x, J_y, \phi_x, \phi_y) \]

In the general case, there are no invariants and there is not any guarantee of stable motion for all initial conditions.

Leads to confined operating space called “dynamic aperture”.
Steepness


Integrable system with a periodic perturbation:

\[ H = H_0(J) + \epsilon H_1(J, \psi, t) \quad H_1 \text{ periodic in } t \text{ over } 2\pi \text{ interval} \]

actions \( J_1, J_2, \ldots \) and angles \( \psi_1, \psi_2, \ldots \)

Then action-invariants are bounded by:

\[ |J(t) - J(0)| < \epsilon^b \]

over the time interval:

\[ T = \frac{1}{\epsilon} \exp \left( \frac{1}{\epsilon^a} \right) \]

If \( H(I) \) meets a certain criteria called steepness.

1D steepness is given by:

\[ \left| \frac{dH_0(J)}{dJ} \right| \geq g > 0 \]

Multi-dimensional similar idea, but not simple
Collective-Instabilities & Landau Damping

In addition to the single-particle effects from external field, there are **collective effects**: The beam itself generates EM fields, which interact with the environment and then back on the beam. Every beam mode represents a feedback loop that can become unstable.

These instabilities can be suppressed by either:

1. **External damping system** – an external kickers system which needs to be activated with the requisite gain and bandwidth.

2. **Landau damping** – an active decoherent effects from the tune-spread of the betatron oscillation. **Nonlinear focusing** has an inherently stabilizing effect on the beam!
New Paradigm – Nonlinear Integrable Optics

Instead of trying to make a perfect linear system and then compensate all the defects, start from something more robust!

The accelerators optics should be:

2D Integrable – no resonances or dynamical chaos.

Widely Stable – no scattering trajectories or separatrices.

Strongly Nonlinear – robust to perturbations, external & collective.
Several Methods for Nonlinear Integrable Optics

- General dynamical systems
  - Special case

- Integrable systems
  - Applied to beam physics

Nonlinear integrable optics

- Thin nonlinear element with specific kick function in symmetric lattice with \( \sin(\mu) = \pm 1 \)
  - "McM paradigm"
  - Examples:
    - Round beams in colliders
    - McMillan electron lens

- Thick nonlinear element shaped with amplitude function in symmetric lattice with \( \cos(\mu) = \pm 1 \)
  - "DN paradigm"
  - Known 2-dimensional cases
  - Examples:
    - Axially-symmetric kicks in solenoid, like in an electron lens
    - Special multipole magnet
FAST/IOTA Facility
FAST: Fermilab Accelerator Science and Technology
- 300 MeV electron superconducting linac
- 2.5 MeV proton normal-conducting RFQ (early 2020)
- IOTA ring for beam physics experiments
  - To be operated with either protons or electrons

IOTA: Integrable Optics Test Accelerator
## IOTA Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal kinetic energy</td>
<td>$e^-: 150$ MeV, $p+: 2.5$ MeV</td>
</tr>
<tr>
<td>Nominal intensity</td>
<td>$e^-: 1 \times 10^9$, $p+: 1 \times 10^{11}$</td>
</tr>
<tr>
<td>Circumference</td>
<td>40 m</td>
</tr>
<tr>
<td>Bending dipole field</td>
<td>0.7 T</td>
</tr>
<tr>
<td>Beam pipe aperture</td>
<td>50 mm dia.</td>
</tr>
<tr>
<td>Maximum b-function $(x,y)$</td>
<td>12, 5 m</td>
</tr>
<tr>
<td>Momentum compaction</td>
<td>$-0.02 \div 0.1$</td>
</tr>
<tr>
<td>Betatron tune (integer)</td>
<td>$3 \div 5$</td>
</tr>
<tr>
<td>Betatron tune chromaticity</td>
<td>$-15 \div 0$</td>
</tr>
<tr>
<td>Transverse emittance r.m.s.</td>
<td>$e^-: 0.04 \mu m$, $p+: 2 \mu m$</td>
</tr>
<tr>
<td>SR damping time</td>
<td>0.6s ($5 \times 10^6$ turns)</td>
</tr>
<tr>
<td>RF $V,f,q$</td>
<td>$e^-: 1$ kV, 30 MHz, 4</td>
</tr>
<tr>
<td>Synchrotron tune</td>
<td>$e^-: 2 \times 10^{-4} \div 5 \times 10^{-4}$</td>
</tr>
<tr>
<td>Bunch length, momentum spread</td>
<td>$e^-: 12$ cm, $1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>Beam lifetime</td>
<td>$e^-: 1$ hour, $p+: 1$ min</td>
</tr>
</tbody>
</table>
Integrable Optics Test Accelerator

IOTA RING

Injection

Nonlinear Magnetic Inserts

RF Cavity

Optical Stochastic Cooling

Electron Lens

Electrostatic BPMs (position, turn-by-turn)
Sync. light monitors (position and shape)
Combined dipole and skew correctors
Injection bump vertical correctors

Modular, Flexible, Cost-effective Accelerator
Nonlinear Integrable Optics Experiments:
1. Nonlinear Elliptic Magnet for Danilov-Nagaitsev Type
2. Octupole String for Quasi-Integrable Henon-Heiles Type
3. Electron Lens for Integrable Optics
   - Polar-Coordinate Potential
   - McMillan-Type Thin Nonlinear Kick

Other FAST/IOTA Experiments:
• Quantum Effects in Single-Electron Storage
• Optical Stochastic Cooling for Bright Electron Beams
• Electron Column/Lens Space-charge Compensation
• FAST Electron Linac as ILC prototype
• Inverse Compton Scattering X-Ray Source
Nonlinear Magnets for Integrable Optics
Remove Time-Dependence: H becomes invariant

Continuous focusing linear accelerator with equal focusing in horizontal and vertical, then add some potential \( V \):

\[
H = \frac{p_x}{2} + \frac{p_y}{2} + K(s) \left( \frac{x^2}{2} + \frac{y^2}{2} \right) + V(x, y, s)
\]

Transform to normalized coordinates:

\[
x_N(s) = \frac{x(s)}{\sqrt{\beta_x(s)}} \quad p_{xN}(s) = \sqrt{\beta_x(s)}p_x(s) - \frac{\beta_x'(s)}{2\sqrt{\beta_x(s)}}x(s) \quad \psi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')}
\]

\[
H_N = \frac{p_{xN}}{2} + \frac{p_{yN}}{2} + \frac{x_N^2}{2} + \frac{y_N^2}{2} + \beta(\psi) \cdot V[\sqrt{\beta(\psi)}x_N, \sqrt{\beta(\psi)}y_N, s(\psi)]
\]

Chose \( V \) to remove all time-dependence (s, \( \psi \) dependence):

\[
H_N = \frac{p_{xN}}{2} + \frac{p_{yN}}{2} + \frac{x_N^2}{2} + \frac{y_N^2}{2} + U(x_N, y_N)
\]

The nonlinear potential \( V \) varies longitudinally with beam size.
Remove Time-Dependence: H becomes invariant

But we started with a continuous focusing linear accelerator with equal horizontal and vertical optics…

Instead, make a conventional linear accelerator but with $n\pi$ phase-advance and matched beta functions, followed by a nonlinear insert:

In the **nonlinear insert**, the potentials scales with the beam.

In the **T-insert**, the beam oscillates linearly and the betatron phase is the same at start and end, in both planes.

$$H_N = U(x_N, y_N) + \frac{p_{xN}}{2} + \frac{p_{yN}}{2} + \frac{x_N^2}{2} + \frac{y_N^2}{2}$$
If we prepare a generic nonlinear potential this way, for example octupoles, there will generally be only invariant of motion. We may have removed time-dependent resonances, but there will still be coupling resonances between the transverse degrees of freedom. We term this **quasi-integrable** or **Henon-Heiles-type**.

\[ U(x, y) = \kappa \left( \frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2} \right) \]
Performance of Octupole Henon-Heiles

What is the tune-spread within the dynamic aperture?

While Dynamic Aperture is limited, the attainable tune spread is large ~0.03 – compare to ~0.001 created by LHC octupoles

S. Antipov, S. Nagaitsev, A. Valishev,
JINST 12 (2017) no.04, P04008
Second-Invariant

If we know there is a second invariant \( I \) for Hamiltonian \( H \), it is easy to verify it: \([H, I] = 0\)

So we can assume a generic form of \( I \), impose \([H, I] = 0\), and see we can find a usable solution:

\[
I = A(x, y)p_x^2 + B(x, y)p_xp_y + C(x, y)p_y^2 + D(x, y)
\]

wlog, \( A(x, y) = ay^2 + c, \quad B(x, y) = -2axy, \quad C(x, y) = ax^2 \)

a=1: \( A(x, y) = y^2 + c, \quad B(x, y) = -2xy, \quad C(x, y) = x^2 \)

\[
xy \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) U + (y^2 - x^2 + c^2) \frac{\partial^2}{\partial x \partial y} U + 3y \frac{\partial}{\partial x} U - 3x \frac{\partial}{\partial y} U = 0
\]
Second-Invariant, elliptic

Bertrand-Darboux Equation:

\[ xy \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) U + (y^2 - x^2 + c^2) \frac{\partial^2}{\partial x \partial y} U + 3y \frac{\partial}{\partial x} U - 3x \frac{\partial}{\partial y} U = 0 \]

The solution emerges from elliptic coordinates:

\[ \xi = \frac{\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2}}{2c} \]
\[ \eta = \frac{\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2}}{2c} \]

\[ U(x, y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \]

\[ I(x, y, p_x, p_y) = (xp_y - yp_x)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2} \]
Second-Invariant with magnets

\[ U(x, y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \]

If we want a potential we can implement with magnets, we have to further impose Laplace’s Equation:
\[ \nabla^2 U(x, y) = 0 \]

\[ f(\xi) = \frac{1}{2} f_1(\xi) + f_2(\xi) \quad g(\xi) = \frac{1}{2} g_1(\xi) + g_2(\xi) \]

**Linear term:**  
\[ f_1(\xi) = c^2 \xi^2 (\xi^2 - 1) \quad g_1(\eta) = c^2 \eta^2 (1 - \eta^2) \]

**Nonlinear term:**  
\[ f_2(\xi) = \xi \sqrt{\xi^2 - 1} [d + t \cdot \text{acosh}(\xi)] \]
\[ g_2(\xi) = \eta \sqrt{1 - \eta^2} [b + t \cdot \text{acos}(\eta)] \]

If there is any dipole-term, this will immediately generate dispersion, so we should further impose:  
\[ d = 0, \quad b = -t\pi/2 \]
Nonlinear Elliptic Potential

\[ \xi = \frac{\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2}}{2c} \]

\[ \eta = \frac{\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2}}{2c} \]

\[ f_2(\xi) = t\xi \sqrt{\xi^2 - 1} \text{acosh}(\xi) \]

\[ g_2(\xi) = t\eta \sqrt{1 - \eta^2} \left[ -\frac{\pi}{2} + \text{acos}(\eta) \right] \]

\[ U(x,y) = -\frac{t}{c^2} \text{Im} \left( (x + iy)^2 + \frac{2}{3c^2} (x + iy)^4 + \frac{8}{15c^4} (x + iy)^6 + \ldots \right) \]
Segmented Elliptic Magnet
IOTA Lattice with NL Insert

Nonlinear insert:
Landau Damping vs. Antidamper

Emulate a collective instability with an anti-damper: \[
\frac{\Delta p_{\perp}}{p_0} = g\langle x \rangle
\]

Nonlinear element reduces max centroid oscillation by factor of 50 and reduces particle loss by factor 100.
Damping Performance vs. Octupole

For an initially displaced beam:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I$</td>
<td>0.8225</td>
<td>mA</td>
</tr>
<tr>
<td>$dQ_{SC}$</td>
<td>-0.03</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>8</td>
<td>mm-mrad</td>
</tr>
<tr>
<td>$K$</td>
<td>2.5</td>
<td>MeV</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>100</td>
<td>$\mu m$</td>
</tr>
<tr>
<td>Nonlinear Magnet Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>0.4</td>
<td>-</td>
</tr>
<tr>
<td>$c$</td>
<td>0.1</td>
<td>$m^{1/2}$</td>
</tr>
<tr>
<td>$\psi_{nll}$</td>
<td>0.3</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>Octupole Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_3$</td>
<td>35000</td>
<td>$m^{-4}$</td>
</tr>
<tr>
<td>$L$</td>
<td>1.8</td>
<td>$m$</td>
</tr>
<tr>
<td>$B_\rho$</td>
<td>0.23</td>
<td>$Tm$</td>
</tr>
</tbody>
</table>

It is possible to achieve comparable rates of damping with an octupole. Octupole produces larger fluctuations. Requires high pole-tip magnetic fields.
Damping Animation

16 mm-mrad
500 μm offset
20 mm-mrad
No offset

Chirs Hall, Radiasoft, IOTA 2017
Nonlinear Electron Lens for Integrable Optics
Second-Invariant, Polar coordinates

\[ I = A(x, y)p_x^2 + B(x, y)p_xp_y + C(x, y)p_y^2 + D(x, y) \quad [H, I] = 0 \]

\[ A(x, y) = ay^2 + c, \quad B(x, y) = -2axy, \quad C(x, y) = ax^2 \]

\( c=0: \quad H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{r}{2} + f(r) + \frac{g(\theta)}{r^2} \quad I = (xp_y - yp_x)^2 + 2g(\theta) \]

Imposing Laplace’s equation may be too constraining:

\[ f(r) = d \ln(r), \quad \text{or} \quad g(\theta) = b \sin(2\theta) + t \cos(2\theta) \]

So look for solutions without Laplace’s equation.

Use an electron lens to make an arbitrary circular potential \( f(r) \).
Electron Lens

1. Intense electron beam created at high voltage.
2. E-beam profile shaped by electrodes.
3. E-beam transported with strong solenoid.
4. Electrons collected after single pass.

Electron gun
1 A @ 5 kV

5-keV electron beam

Main solenoid
0.33 T field
0.7 m length

Collector
20 kW

Stancari

Integrable Particle Dynamics in Accelerators | Jan 2019 USPAS
Round, Thick-Lens Kick

Following the previous recipe, the electron beam should be tapered to match the proton beam size. But this may be difficult.

Instead, use the solenoid to maintain constant beta functions of the proton beam and then apply a constant electron beam.

The electron beam distribution can be any round distribution. e.g. Gaussian profile
McMillan-Type Thin Kick

We can also use the electron lens to implement the McMillan Map:

\[
\begin{pmatrix}
0 & \beta & 0 & 0 \\
-1/\beta & 0 & 0 & 0 \\
0 & 0 & 0 & \beta \\
0 & 0 & -1/\beta & 0
\end{pmatrix}
\]

**linear arc:**

**Electron Lens Beam Profile**

**current density**

\[ j(r) = \frac{j_0 a^4}{(r^2 + a^2)^2} \]

**transverse kick**

\[ \theta(r) = \frac{k_e a^2 r}{r^2 + a^2} \]

Integral of motion:

\[ I_x = b x^2 p^2 + x^2 + p^2 - a x p \]

\[ x_i = p_{i-1} \]

\[ p_i = -x_{i-1} + \frac{a p_{i-1}}{b p_{i-1}^2 + 1} \]

The electron lens should be a thin kick, \( \beta > L \)
Implementation in IOTA Ring

- The lattice is made of 2 main building blocks
  - an axially symmetric, linear arc with specified phase advance, equivalent to a thin lens (“T-insert”)
  - a nonlinear section with equal beta functions and
    - nonlinear magnet or
    - thin, round McMillan-type kick (electron lens option #1) or
    - any axially symmetric kick in solenoid (electron lens option #2)

linear arc:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-k & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -k & 1 \\
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
0 & \beta & 0 & 0 & 0 \\
-1/\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 \\
0 & 0 & -1/\beta & 0 & 0 \\
\end{pmatrix}
\]

for McMillan kicks
Implementation in IOTA Ring

Use the electromagnetic field generated by the electron distribution to provide the desired nonlinear field.
Linear focusing strength on axis \( \sim 1/m \):

\[
k_e = 2\pi \frac{j_0 L (1 \pm \beta_e \beta_z)}{(B\rho) \beta_e \beta_z c^2} \left( \frac{1}{4\pi \varepsilon_0} \right).
\]

1. Axially symmetric thin kick of McMillan type

current density \( j(r) = \frac{j_0 a^4}{(r^2 + a^2)^2} \)

transverse kick \( \theta(r) = \frac{k_e a^2 r}{r^2 + a^2} \)

achievable tune spread \( \sim \frac{\beta k_e}{4\pi} \)

Larger tune spreads in IOTA
More sensitive to kick shape

2. Axially symmetric kick in long solenoid

Any axially-symmetric current distribution

\[
\sim \frac{L}{2\pi \beta} = \frac{L B_z}{4\pi (B\rho)}
\]

Smaller tune spreads in IOTA
More robust
Simulation HL-LHC with Electron Lens Landau Damping

octupoles

\[ \text{DA}_{\text{min}} = 3.7 \sigma \]

E-Lens

\[ \text{DA}_{\text{min}} = 8.0 \sigma \]

Shiltsev PRL 2017
Several Methods for Nonlinear Integrable Optics

General dynamical systems

Integrable systems

Nonlinear integrable optics

Thin nonlinear element with specific kick function in symmetric lattice with \( \sin(\mu) = \pm 1 \) "McM paradigm"

Examples
- Round beams in colliders
- McMillan electron lens

Thick nonlinear element shaped with amplitude function in symmetric lattice with \( \cos(\mu) = \pm 1 \) "DN paradigm"

Examples
- Axially-symmetric kicks in solenoid, like in an electron lens
- Special multipole magnet

Applied to beam physics

Known 2-dimensional cases