Collective Effects and Luminosity

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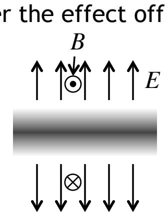
Space Charge

So far, we have not considered the effect that particles in a bunch might have on each other, or on particles in another bunch.

Consider the effect off space charge on the transverse distribution of the beam.



$$\rho(r) = \frac{Ne}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

radial charge density

If we look at the field at a radius r, we have

$$\oint \vec{E} \bullet d\vec{A} = E(2\pi r L) = \frac{Q_{encl}}{\epsilon_0} = \frac{Ne}{\sigma^2}\int_0^r r e^{-r^2/2\sigma^2}\, dr$$

$$= Ne\left(1 - e^{-r^2/2\sigma^2}\right)$$

$$\longrightarrow \vec{E} = \frac{Ne}{2\pi\epsilon_0 r L}\left(1 - e^{-r^2/2\sigma^2}\right)\hat{r}$$

Similarly, Ampere's Law gives

$$\oint \vec{B}\bullet d\vec{l} = 2\pi r B = \mu_0 I_{enclosed} = \mu_0 \frac{Nev}{\sigma^2 L}\int_0^r r e^{-r^2/2\sigma^2}\,dr$$

$$\longrightarrow \vec{B} = \mu_0 \frac{Nev}{2\pi r L}\left(1-e^{-r^2/2\sigma^2}\right)\hat{\theta}$$

$$\longrightarrow \vec{F} = e\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

$= -\hat{r}$

$$= \frac{Ne^2}{2\pi L}\left(1-e^{-r^2/2\sigma^2}\right)\left(\frac{1}{\epsilon_0}\hat{r} + v^2\mu_0\left(\hat{s}\times\hat{\theta}\right)\right)$$

$$= \frac{1}{\epsilon_0}\left(\epsilon_0\mu_0\right) = \frac{1}{\epsilon_0}\frac{1}{c^2}$$

$$= \hat{r}\frac{Ne^2}{2\pi r L\epsilon_0}\left(1-e^{-r^2/2\sigma^2}\right)\left(1-\beta^2\right)$$

$$= \hat{r}\frac{ne^2}{2\pi r\epsilon_0\gamma^2}\left(1-e^{-r^2/2\sigma^2}\right);\quad \mathrm{n} \equiv \frac{N}{L} = \frac{dN}{ds}$$ Linear charge density

We can break this into components in x and y

$$F_x = |F|\frac{x}{r}$$

$$= \frac{ne^2}{2\pi\epsilon_0\gamma^2}\frac{x}{r^2}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$= \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{x}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

$$F_y = \frac{ne^2}{2\pi r\epsilon_0\gamma^2}\frac{y}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$

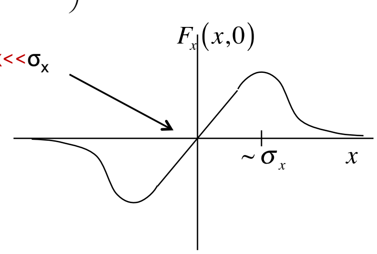
Non-linear and coupled ➔ ouch! but for $x \ll \sigma_x$

$$\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right) \approx \frac{\left(x^2+y^2\right)}{2\sigma^2}$$

$$\longrightarrow F_x \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}x$$

$$F_y \approx \frac{ne^2}{4\pi\sigma^2\epsilon_0\gamma^2}y$$

~linear and decoupled



$$x'' = \frac{F_x}{vp} = \frac{F_x}{\beta^2 \gamma m c^2}$$
$$\approx \frac{e^2}{4\pi\sigma^2\epsilon_0\beta^2\gamma^3} nx$$
$$= \frac{r_0}{\beta^2\gamma^3\sigma^2} nx; \quad r_0 \equiv \frac{e^2}{4\pi\epsilon_0 m_0 c^2}$$

"classical radius" $= 1.53 \times 10^{-18}$ m for protons

This looks like a distributed defocusing quad of strength $\frac{d\left(\frac{1}{f}\right)}{ds} \equiv k = -\frac{nr_0}{\beta^2\gamma^3\sigma^2}$

so the total tuneshift is $\Delta\nu_x = \frac{1}{4\pi}\oint k\beta_x(s)ds$

$$= -\frac{r_0}{4\pi\beta^2\gamma^3}\oint n\frac{\beta_x(s)}{\sigma_x^2}ds \quad = \frac{1}{\epsilon_x}$$
$$= -\frac{r_0}{4\pi\beta^2\gamma^3}\frac{NB}{\epsilon_x}; \quad \mathrm{B} \equiv \frac{n_{peak}}{\langle n \rangle}$$

"Bunching factor"

$$= -\frac{NBr_0}{4\pi\beta\gamma^2 L}\frac{L}{\underbrace{(\beta\gamma\epsilon_x)}_{\equiv \epsilon_{x,N}}}$$
$$= -\frac{NBr_0}{4\pi\beta\gamma^2\epsilon_{x,N}}$$

Maximum tuneshift for particles near core of beam

Example: Fermilab Booster@Injection

$K = 400$ MeV

$\mathrm{N} = 5 \times 10^{12}$

$\epsilon_N = 2\ \pi$-mm-mr

B=1 (unbunched beam)

$$\Delta_\nu = -\frac{Nr_0}{4\pi\beta\gamma^2\epsilon_N} = -.247$$

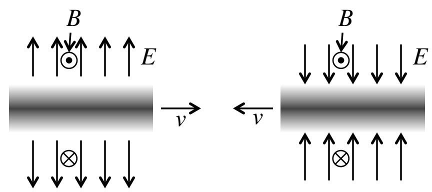
This is pretty large, but because this is a rapid cycling machine, it is less sensitive to resonances

Because this affects individual particles, it's referred to as an "incoherent tune shift", which results in a tune spread. There is also a "coherent tune shift", caused by images charges in the walls of the beam pipe and/or magnets, which affects the entire bunch more or less equally.

This is an important effect, but beyond the scope of this lecture.

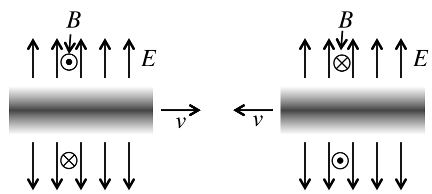
Beam-beam Interaction

If two *oppositely* charged bunches pass through each other…



Both E and B fields are *attractive* to the particles in the other bunch

If two bunches with the *same* sign pass through each other…



Both E and B fields are *repulsive* to the particles in the other bunch

In either case, the forces add

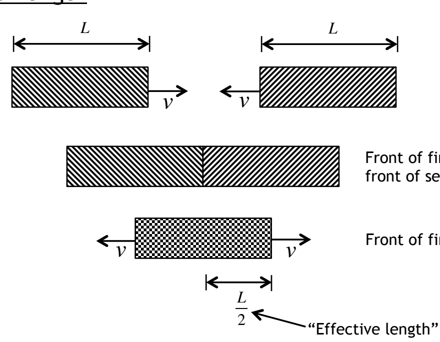
$$\vec{F} = -\hat{r}\frac{e^2}{2\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)\overset{\approx 2}{\left(1+\beta^2\right)}$$

$$\approx -\hat{r}\frac{e^2}{\pi\epsilon_0 r}\frac{N}{L}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$x'' = \frac{F_x}{vp};\quad y'' = \frac{F_y}{vp} \quad \Longrightarrow \quad \Delta x' = \frac{F_x}{vp}\Delta s;\quad \Delta y' = \frac{F_y}{vp}\Delta s$$

Integrate…

<u>Effective Length</u>



Front of first bunch encounters front of second bunch

Front of first bunch exits second bunch.

$$\Delta x' = \frac{F_x}{vp}\Delta s = \frac{F_x}{vp}\left(\frac{L}{2}\right)$$
$$= -\frac{N_b e^2}{2\pi\epsilon_0 r\gamma\underbrace{\beta^2}_{\beta\approx 1} mc^2}\frac{x}{r}\left(1-e^{-r^2/2\sigma^2}\right)$$
$$\approx -\frac{2N_b r_0}{\gamma}\frac{x}{r^2}\left(1-e^{-r^2/2\sigma^2}\right)$$
$$= -\frac{2N_b r_0}{\gamma}\frac{x}{\left(x^2+y^2\right)}\left(1-e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right)$$
$$\approx -\frac{N_b r_0}{\gamma\sigma^2}x = -\frac{1}{f_{eff}}x$$

Small x and y

$$\Delta y' \approx -\frac{N_b r_0}{\gamma\sigma^2}y = -\frac{1}{f_{eff}}y$$

$$\longrightarrow \Delta\nu = \frac{N_b r_0 \beta^*}{f_{eff}} = \frac{N_b r_0}{4\pi}\frac{\beta^*}{\gamma\sigma^2}$$

Maximum tuneshift for particles near center of bunch

$$= \frac{N_b r_0}{4\pi\epsilon_N}$$

normalized emittance

$$\boxed{\equiv \xi}$$

“Tuneshift Parameter”

Luminosity and Tuneshift

The total tuneshift will ultimately limit the performance of any collider, by driving the beam onto an unstable resonance. Values of on the order ~.02 are typically the limit. However, we have seen the somewhat surprising result that the tuneshift

$$\xi = \frac{N_b r_0}{2\pi\epsilon\gamma}$$

does not depend on β^*, but only on

$$\frac{N_b}{\epsilon} \equiv \text{"brightness"}$$

For a collider, we have

$$\mathcal{L} = \frac{f n_b N_b^2}{4\pi\sigma^2} = \frac{f n_b N_b^2}{4\pi\left(\frac{\beta^*\epsilon_N}{\gamma}\right)} = \frac{f n_b N_b \gamma}{r_0\beta^*}\left(\frac{r_0}{4\pi}\frac{N_b}{\epsilon_N}\right)$$
$$= f\frac{n_b N_b \gamma}{r_0\beta^*}\xi$$

We assume we will run the collider at the “tuneshift limit”, in which case we can increase luminosity by

- Making β^* as small as possible
- Increasing N_b and ε proportionally.