Floquet Transformations and Harmonic Resonances

Non-linear Perturbations

- In our earlier lectures, we found the general equations of motion

$$x'' = -\frac{B_y}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2 + \frac{\rho + x}{\rho^2}$$

$$y'' = \frac{B_x}{(B\rho)}\left(1+\frac{x}{\rho}\right)^2$$

- We initially considered only the linear fields, but now we will bundle all additional terms into ΔB
  - non-linear plus linear field errors
- We see that if we keep the lowest order term in ΔB, we have

$$B_y = B_0 + B'x + \Delta B_y(x,s)$$

$$B_x = \underbrace{\qquad B'y}_{\text{This part gave us the Hill's equation}} + \Delta B_x(y,s)$$

$$\boxed{\begin{aligned} x'' + K(s)x &= -\frac{1}{(B\rho)}\Delta B_y(x,s) \\ y'' + K(s)y &= \frac{1}{(B\rho)}\Delta B_x(y,s) \end{aligned}}$$

Floquet Transformation

- Evaluating these perturbed equations can be very complicated, so we will seek a transformation which will simplify things
- Our general equation of Motion is

$$x(s) = A\sqrt{\beta(s)}\cos(\psi(s) + \delta)$$

- This looks quite a bit like a harmonic oscillator, so not surprisingly there is a transformation which looks *exactly* like harmonic oscillations

$$\xi(s) = \frac{x}{\sqrt{\beta}}$$

$$\phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int \frac{1}{\beta} ds \Rightarrow \frac{d\phi}{ds} = \frac{1}{\nu\beta}$$

Plugging back into the Equation

$$x = \sqrt{\beta}\xi$$

$$x' = \frac{1}{2}\frac{1}{\sqrt{\beta}}\beta'\xi + \beta^{1/2}\frac{d\xi}{d\phi}\frac{d\phi}{ds} = -\alpha\frac{1}{\sqrt{\beta}}\xi + \frac{1}{\nu\sqrt{\beta}}\dot{\xi}$$

$$= \frac{1}{\nu\sqrt{\beta}}\left(\dot{\xi} - \alpha\nu\xi\right)$$

$$x'' = \frac{\alpha}{\nu\beta^{3/2}}\left(\dot{\xi} + \alpha\nu\xi\right) + \frac{1}{\nu\sqrt{\beta}}\left(\frac{\ddot{\xi}}{\nu\beta} - \alpha'\nu\xi - \frac{\alpha\dot{\xi}}{\beta}\right) =$$

$$= \frac{\ddot{\xi} - \nu^2\left(\alpha^2\xi + \beta\alpha'\right)\xi}{\nu^2\beta^{3/2}}$$

So our differential equation becomes

$$x'' + K(s)x = \frac{\ddot{\xi} - \nu^2\left(\alpha^2 + \beta\alpha'\right)\xi}{\nu^2\beta^{3/2}} + K(s)\beta^{1/2}\xi$$

$$= \frac{\ddot{\xi} - \nu^2\left(\alpha^2 + \beta\alpha' - \beta^2 K\right)\xi}{\nu^2\beta^{3/2}} = -\frac{\Delta B}{(B\rho)}$$

- We showed a few lectures back that

$$\psi' = \frac{k}{w^2(s)}$$

$$w''(s) + K(s)w(s) - \frac{k}{w^3(s)} = 0$$

$$\Rightarrow K\beta^2 - \beta\alpha' - \alpha^2 = 1$$

- So our rather messy equation simplifies

$$\frac{\ddot{\xi} - \nu^2\left(\alpha^2 + \beta\alpha' - \beta^2 K\right)\xi}{\nu^2\beta^{3/2}} = -\frac{\Delta B}{(B\rho)}$$

$$\Rightarrow \boxed{\ddot{\xi} + \nu^2\xi = -\nu^2\beta^{3/2}\frac{\Delta B}{(B\rho)}}$$

Understanding Floquet Coordinates

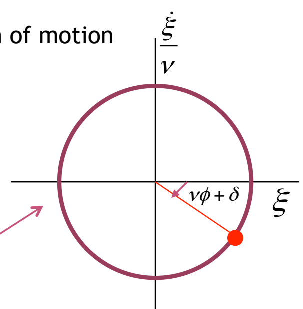
- In the absence of nonlinear terms, our equation of motion is simply that of a harmonic oscillator

$$\ddot{\xi}(\phi) + \nu^2\xi(\phi) = 0$$

and we write down the solution

$$\xi(\phi) = a\cos(\nu\phi + \delta)$$

$$\dot{\xi}(\phi) = -a\nu\sin(\nu\phi + \delta)$$



- Thus, motion is a circle in the $\left(\xi, \frac{\dot{\xi}}{\nu}\right)$ plane
- Using our standard formalism, we can express this as

$$\xi(\phi) = \xi_0\cos(\nu\phi) + \frac{\dot{\xi}_0}{\nu}\sin(\nu\phi)$$
$$\dot{\xi}(\phi) = -\xi_0\nu\sin(\nu\phi) + \dot{\xi}_0\cos(\nu\phi)$$
$$\Rightarrow \begin{pmatrix} \xi(\phi) \\ \dot{\xi}(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\nu\phi) & \tilde{\beta}\sin(\nu\phi) \\ -\frac{1}{\tilde{\beta}}\sin(\nu\phi) & \cos(\nu\phi) \end{pmatrix} \begin{pmatrix} \xi_0 \\ \dot{\xi}_0 \end{pmatrix}; \text{ where } \tilde{\beta} \equiv \frac{1}{\nu}$$

- A common mistake is to view ϕ as the phase angle of the oscillation.
  - $\nu\varphi$ the phase angle of the oscillation
  - φ advances by 2π in one revolution, so it's *related* (but NOT equal to!) the angle around the ring.

$$\text{Note: } x_{\max}^2 = \beta\varepsilon = \beta\xi_{\max}^2 = \beta a^2 \Rightarrow a^2 = \epsilon$$

unnormalized!

Perturbations

- In general, resonant growth will occur if the perturbation has a component at the same frequency as the unperturbed oscillation; that is if

$$\Delta B(\xi,\phi) = ae^{i\nu\phi} + (\ldots) \Rightarrow \text{resonance!}$$

- We will expand our magnetic errors at one point in ϕ as

$$\Delta B(x) \equiv b_0 + b_1 x + b_2 x^2 + b_3 x^3 \ldots; b_n \equiv \frac{1}{n!}\frac{\partial^n B}{\partial x^n}\Big|_{x=y=0}$$

$$-\frac{\nu^2\beta^{3/2}\Delta B}{(B\rho)} = -\frac{\nu^2}{(B\rho)}\left(\beta^{3/2}b_0 + \beta^{4/2}b_1\xi + \beta^{5/2}b_2\xi^2 + \ldots\right)$$

$$\ddot{\xi} + \nu^2\xi = -\frac{\nu^2}{(B\rho)}\sum_{n=0}^{\infty}\beta^{(n+3)/2}b_n\xi^n$$

- But in general, b_n is a function of φ, as is β, so we bundle all the dependence into harmonics of φ $$\frac{1}{(B\rho)}\beta^{(n+3)/2}b_n = \sum_{m=-\infty}^{\infty}C_{m,n}e^{im\phi}$$
- So the equation associated with the n^th driving term becomes

$$\ddot{\xi} + \nu^2\xi = -\nu^2\sum_{k=-\infty}^{\infty}C_{m,n}\xi^n e^{im\phi}$$

Remember!
ξ, β, and b_n are all functions of (only) φ

Calculating Driving Terms

- We can calculate the coefficients in the usual way with

$$C_{m,n} = \frac{1}{(B\rho)}\frac{1}{2\pi}\int_0^{2\pi}\beta^{(n+3)/2}b_n e^{-im\phi}d\phi$$

- But we generally know things as functions of *s*, so we use $d\phi = \frac{1}{\nu}d\psi = \frac{1}{\nu}\frac{d\psi}{ds}ds = \frac{1}{\nu\beta}ds$ to get

$$\boxed{C_{m,n} = \frac{1}{(B\rho)}\frac{1}{2\pi\nu}\oint\beta^{(n+1)/2}(s)b_n(s)e^{-im\phi}ds}$$

Where (for a change) we have explicitly shown the *s* dependent terms.

- We're going to assume small perturbations, so we can approximate β with the solution to the homogeneous equation

$$\xi(\phi) \approx a\cos(\nu\phi);\ \text{(define starting point so } \delta = 0)$$

$$\xi^n = a^n\cos^n(\nu\phi) = a^n\frac{1}{2^n}\sum_{\substack{k=-n\\ \Delta k=2}}^{n}\begin{pmatrix} n \\ \frac{n-k}{2}\end{pmatrix}e^{i\nu k\phi};\ \text{where}\begin{pmatrix} i \\ j\end{pmatrix} \equiv \frac{i!}{j!(i-j)!}$$

- Plugging this in, we can write the nth driving term as

$$-\nu^2\left(\frac{a}{2}\right)^n \sum_{\substack{k=-n \\ \Delta k=2}}^{n} \binom{n}{\frac{n-k}{2}} \sum_{m=-\infty}^{\infty} C_{m,n} e^{i(m+\nu k)\phi}$$

- We see that a resonance will occur whenever

$$\begin{aligned} m+\nu k &= \pm\nu \\ \nu(1\mp k) &= \pm m \end{aligned} \quad \text{where} \quad \begin{aligned} &-\infty < m < \infty \\ &-n \le k \le n \quad (\Delta k = 2) \end{aligned}$$

- Since *m* and *k* can have either sign, we can cover all possible combinations by writing

$$\nu_{\text{resonant}} = \frac{m}{1-k}$$

Types of Resonances

Magnet Type	*n*	*k*	Order |*1-k*|	Resonant tunes *ν=m/(1-k)*	Fractional Tune at Instability
Dipole	0	0	1	*m*	*0,1*
Quadrupole	1	1	0	*none (tune shift)*	-
	1	-1	2	*m/2*	*0,1/2,1*
Sextupole	2	2	1	*m*	*0,1*
	2	0	1	*m*	*0,1*
	2	-2	3	*m/3*	*0,1/3,2/3,1*
Octupole	3	3	2	*m/2*	*0,1/2,1*
	3	1	0	*None*	-
	3	-1	2	*m/2*	*0,1/2,1*
	3	-3	4	*m/4*	*0,1/4,1/2,3/4,1*

Effect of Periodicity

- If our ring is *perfectly* periodic (never quite true), with a period N, then we can express our driving term as

$$C_{n,m} = \int_0^{2\pi} f(\phi)e^{-im\phi}d\phi = \int_0^{\frac{2\pi}{N}} f(\phi)e^{-im\phi}d\phi + \int_0^{\frac{2\pi}{N}} f(\phi+\frac{2\pi}{N})e^{-im\left(\phi+\frac{2\pi}{N}\right)}d\phi + \int_0^{\frac{2\pi}{N}} f(\phi+2\frac{2\pi}{N})e^{-im\left(\phi+2\frac{2\pi}{N}\right)}d\phi + \ldots$$

$$= \left(\int_0^{\frac{2\pi}{N}} f(\phi)e^{-im\phi}d\phi\right)\sum_{l=0}^{N-1} e^{-im\left(l\frac{2\pi}{N}\right)}$$

- Where we have invoked the periodicity as $f\left(\phi+\frac{2\pi}{N}\right)=f(\phi)$
- Clearly, if m is any integer multiple of N, then all values are *1*. Otherwise, the sum describes a closed path in the complex plane, which adds to zero, so

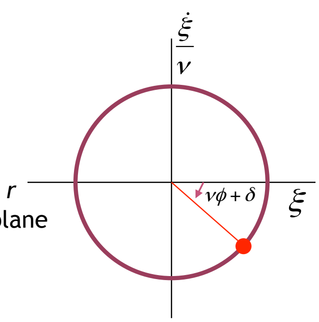
$$\boxed{\begin{aligned} C_{n,m} &= N\int_0^{\frac{2\pi}{N}} f(\phi)e^{-im\phi}d\phi \quad &&\text{if } m=\pm jN \\ &= 0 &&\text{if } m \neq \pm jN \end{aligned}}$$

- That is, we are only sensitive to terms where m is a multiple of the periodicity.
  - This reduces the effect of the periodic non-linearities

Resonant Behavior

- Remember that our unperturbed motion is just

$$\xi(\phi) = a\cos(\nu\phi+\delta)$$
$$\dot\xi(\phi) = -a\nu\sin(\nu\phi+\delta)$$



- A resonance will modify the shape and size of this trajectory, so we replace *a* with a variable *r* and we can now express the postion in the $r\xi$ plane

$$\xi(\phi) = r\cos(\nu\phi+\delta) = r\cos\theta$$
$$\dot\xi(\phi) = -r\nu\sin(\nu\phi+\delta) = -r\nu\sin\theta$$
$$\theta = \tan^{-1}\frac{\dot\xi}{\nu\xi}$$

- We express r^2 in terms of our variables and we have

$$r^2 = \xi^2 + \left(\frac{\dot\xi}{\nu}\right)^2$$

$$\frac{d}{d\phi}r^2 = 2\xi\dot\xi + 2\dot\xi\frac{\ddot\xi}{\nu^2} = 2\frac{\dot\xi}{\nu^2}\overbrace{\left(\ddot\xi+\nu^2\xi\right)}^{\text{Plug in } n^{th} \text{ driving term for this}}$$

$$= -2\dot\xi\sum_{m=-\infty}^{\infty} C_{m,n}\xi^n e^{im\phi} = 2\nu\cos^n\theta\sin\theta r^{n+1}\sum_{m=-\infty}^{\infty} C_{m,n}e^{im\phi}$$

Evolution of Angular Variable

- We have

$$\theta = \tan^{-1}\left(-\frac{\dot{\xi}}{\nu\xi}\right) \Rightarrow$$

$$\frac{d\theta}{d\phi} = \frac{1}{1+\left(\frac{\dot{\xi}}{\nu\xi}\right)^2}\left(-\frac{\ddot{\xi}}{\nu\xi}+\frac{\dot{\xi}^2}{\nu\xi^2}\right) = \frac{\nu}{\nu^2\xi^2+\dot{\xi}^2}\left(-\xi\ddot{\xi}+\dot{\xi}^2\right)$$

use

$$\ddot{\xi} = -\nu^2 \sum_{m=-\infty}^{\infty} C_{m,n}\xi^n e^{im\phi} - \nu^2\xi$$

$$= \nu\left(1+\frac{\nu^2\xi^{n+1}\sum_{m=-\infty}^{\infty} C_{m,n}e^{im\phi}}{\nu^2\xi^2+\dot{\xi}^2}\right) = \nu\left(1+\frac{\nu^2\xi^{n+1}\sum_{m=-\infty}^{\infty} C_{m,n}e^{im\phi}}{\nu^2 r^2}\right)$$

$$= \nu\left(1+\cos^{n+1}\theta\, r^{n-1}\sum_{m=-\infty}^{\infty} C_{m,n}e^{im\phi}\right)$$

- These are our general equations to evaluate the effects of particular types of field errors. Remember that our sensitivity to these errors is actually contained in the $C_{m,n}$ coefficients

Example: Third Order Resonance

- Sextupole terms (n=2) can drive a third order resonance

$$\frac{dr^2}{d\phi} = 2\nu r^3\cos^2\theta\sin\theta\sum_{m=-\infty}^{\infty} C_{m,2}e^{im\phi}$$

$$\frac{d\theta}{d\phi} = \nu\left(1+r\cos^3\theta\sum_{m=-\infty}^{\infty} C_{m,2}e^{im\phi}\right)$$

- We will consider one value of $|m|$ at a time

$$C_{m,2}e^{im\phi} = \frac{1}{2\pi\nu}\oint\beta^{3/2}\frac{b_2}{(B\rho)}e^{im(\phi-\phi')}ds$$

- We'll redefine things in terms of all real components by combining the positive and negative m values

$$C_{m,2}e^{im\phi}+C_{-m,2}e^{-im\phi} = \frac{1}{\pi\nu}\oint\beta^{3/2}\frac{b_2}{(B\rho)}\cos(m(\phi-\phi'))ds$$

$$= \frac{1}{\pi\nu}\cos m\phi\oint\beta^{3/2}\frac{b_2}{(B\rho)}\cos m\phi' ds + \frac{1}{\pi\nu}\sin m\phi\oint\beta^{3/2}\frac{b_2}{(B\rho)}\sin m\phi' ds$$

$$\equiv \frac{1}{\pi\nu}\left(A_{m,2}\cos m\phi + B_{m,2}\sin m\phi\right)$$

- So we have define real driving terms

$$A_{m,2} = \oint \beta^{3/2} \frac{b_2}{(B\rho)} \cos m\phi ds$$

$$B_{m,2} = \oint \beta^{3/2} \frac{b_2}{(B\rho)} \sin m\phi ds$$

- So we plug this into the formulas

$$\frac{dr^2}{d\phi} = \frac{2}{\pi} r^3 \cos^2\theta \sin\theta \left(A_{m,2}\cos m\phi + B_{m,2}\sin m\phi\right)$$

$$= \frac{1}{4\pi} r^3 \left(A_{m,2}\left(\sin(\theta+m\phi)+\sin(3\theta+m\phi)+\sin(\theta-m\phi)+\sin(3\theta-m\phi)\right)\right.$$
$$\left. - B_{m,2}\left(\cos(\theta+m\phi)+\cos(3\theta+m\phi)-\cos(\theta-m\phi)-\cos(3\theta-m\phi)\right)\right)$$

- For unperturbed motion $\theta = \psi = \nu\phi$ and we're interested in behavior near the third order resonance, where $\nu \sim m/3$, so

$$3\theta - m\phi \approx 3\left(1 - \frac{m}{3}\right)\phi$$

- All other terms will oscillate rapidly and not lead to resonant behavior

Floquet Transformations and Resonances USPAS, Hampton, VA, Jan. 26-30, 2015 15

- So we're left with

$$\frac{dr^2}{d\phi} = \frac{1}{4\pi} r^3 \left(A_{m,2}\sin(3\theta - m\phi) + B_{m,2}\cos(3\theta - m\phi)\right)$$

- The angular coordinate is given by

$$\frac{d\theta}{d\varphi} = \nu + \frac{1}{\pi} r \cos^3\theta \left(A_{m,2}\cos m\varphi + B_{m,2}\sin m\varphi\right)$$

$$= \nu + \frac{1}{8\pi} r\left(A_{m,2}\cos(3\theta - m\varphi) - B_{m,2}\sin(3\theta - m\varphi)\right) + (\text{terms we don't care about})$$

- We perform *yet another* transformation to the (rotating) coordinate system

$$\tilde{\theta} = \theta - \frac{m}{3}\phi$$

Note: in an unperturbed system, this would just be $\tilde{\theta} = \left(\nu - \frac{m}{3}\right)\phi$

$$\frac{d\tilde{\theta}}{d\phi} = \frac{d\theta}{d\phi} - \frac{m}{3}$$

- We then divide the two differentials to get the behavior of r^2 in this plane

$$\frac{dr^2}{d\tilde{\theta}} = \frac{\frac{dr^2}{d\varphi}}{\frac{d\tilde{\theta}}{d\varphi}} = \frac{\frac{1}{4\pi} r^3\left(A_{m,2}\sin 3\tilde{\theta} + B_{m,2}\cos 3\tilde{\theta}\right)}{\left(\nu - \frac{m}{3}\right) + \frac{1}{8\pi} r\left(A_{m,2}\cos 3\tilde{\theta} - B_{m,2}\sin 3\tilde{\theta}\right)}$$

Floquet Transformations and Resonances USPAS, Hampton, VA, Jan. 26-30, 2015 16

- This equation can be integrated to yield

$$a^2 = r^2 + r^3 \frac{A_{m,2}\cos 3\tilde{\theta} - B\sin 3\tilde{\theta}}{12\pi\left(\nu - \frac{m}{3}\right)} \equiv r^2 + r^3 \frac{A_{m,2}\cos 3\tilde{\theta} - B_{m,2}\sin 3\tilde{\theta}}{12\pi\delta\nu}$$

- A and B are related to the angular distribution of the driving elements around the ring. We can always define our starting point so *B=0*, so let's look at

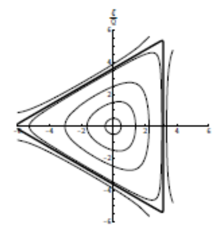
$$a^2 = r^2 + r^3 A_{m,2}\frac{\cos 3\tilde{\theta}}{12\pi\delta\nu}$$

- *a* is an integration constant which is equal to the emittance in the absence of the resonance.
- This is ugly, but let's examine some general features

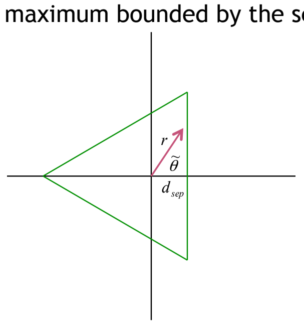
$$\tilde{\theta} = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \quad \Rightarrow r^2 = a^2$$

$$\tilde{\theta} = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \quad \Rightarrow r^2 = r^2_{\min}$$

$$\tilde{\theta} = \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{3} \quad \Rightarrow r^2 = r^2_{\max} \quad \text{no solution for large A}$$



- The separatrix is defined by a triangle. We'd like to solve for the maximum *a* as a function of the driving term *A*. When a corresponds to the maximum bounded by the separatrix, we have that at



$$\tilde{\theta} = \frac{\pi}{6} \Rightarrow r = a_{sep} \Rightarrow d_{sep} = a_{sep}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}a_{sep}$$

Plug this in when the angle =0, and we have

$$a_{sep}^2 = \left(\frac{\sqrt{3}}{2}\right)^2 a_{sep}^2 + \left(\frac{\sqrt{3}}{2}\right)^3 a_{sep}^3 A_{m,2}\frac{\cos 3\tilde{\theta}}{12\pi\delta\nu}$$

$$\Rightarrow \frac{4}{3} = 1 + A_{m,2}\frac{\sqrt{3}}{24\pi\delta\nu}a_{sep}$$

$$\Rightarrow a_{sep} = \frac{8\pi\delta\nu}{\sqrt{3}A_{m,2}} = \sqrt{\varepsilon_{\max}}$$

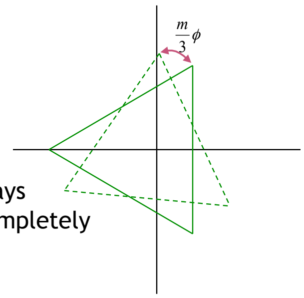
In general

$$\boxed{\begin{aligned} \varepsilon_{\max} &= \frac{64\pi^2\delta\nu^2}{3\left(A_{m,2}^2 + B_{m,2}^2\right)} \qquad & A_{m,2} &= \oint \beta^{3/2}\frac{B''}{2(B\rho)}\cos(3\psi)ds \\ \delta\nu &= \frac{\sqrt{3\varepsilon\left(A_{m,2}^2 + B_{m,2}^2\right)}}{8\pi} \qquad & B_{m,2} &= \oint \beta^{3/2}\frac{B''}{2(B\rho)}\sin(3\psi)ds \end{aligned}}$$

Behavior in Phase Space

- We convert back to our normal Floquet angle

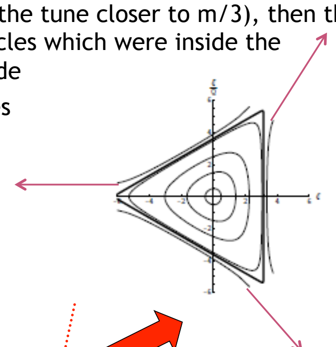
$$a^2 = r^2 + r^3 \frac{A_{m,2}\cos 3\left(\theta - \frac{m}{3}\phi\right) - B_{m,2}\sin 3\left(\theta - \frac{m}{3}\phi\right)}{12\pi\delta\nu}$$

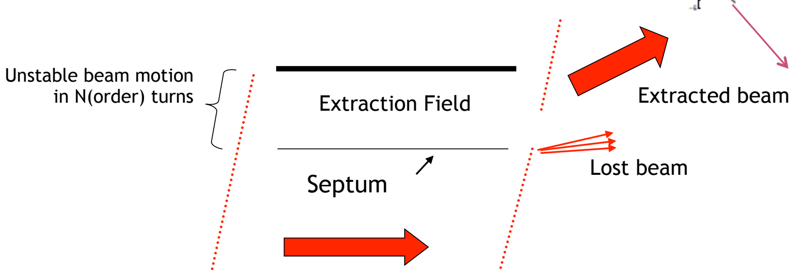


- So as we move around the ring, ϕ advances and the shape will rotate by an amount $(m/3)\phi$
- Since $m/3\text{~}\nu$ is a non-integer, particles must always make three circuits ($\Delta\varphi=6\pi$) before the shape completely rotates at the origin.

Application of Resonance

- If we increase the driving term (or move the tune closer to m/3), then the area of the triangle will shrink, and particles which were inside the separatrix will now find themselves outside
- These will stream out along the asymtotes at the corners.
- These particles can be intercepted by an extraction channel
  - ➔Slow extraction
  - Very common technique





Hamiltonian Approach to Resonances

We now want to define a new coordinate which represents the "flutter" with respect to the average phase advance.

$$\text{"flutter"} = \int_0^s \frac{ds'}{\beta} - 2\pi\nu\frac{s}{C} = \int_0^s \frac{ds'}{\beta} - \nu\frac{s}{R}$$

We define a new coordinate θ, such that

$$\phi = \theta + \text{"flutter"} = \theta + \int_0^s \frac{ds'}{\beta} - \nu\frac{s}{R}$$

We want to transform to new variables θ and I. Try

$$\theta = \phi + \nu\frac{s}{R} - \int^s \frac{ds'}{\beta}$$

$$I = J$$

$$J = \frac{\partial F_2}{\partial \phi}, \quad \theta = \frac{\partial F_2}{\partial I}$$

$$\rightarrow F_2 = I\left(\phi + \nu\frac{s}{R} + \int^s \frac{ds'}{\beta}\right)$$

unperturbed Hamiltonian

$$H_0 = \frac{\nu}{R} I$$

Third Order Resonances Revisited

In the x plane + a sextupole

$$H = \frac{1}{2}p_x^2 + \frac{eB_0}{p_0}x + \frac{1}{2}\frac{eB'}{p_0}x^2 + \frac{1}{6}\frac{eB''}{p_0}x^3$$

$$= H_0 + \frac{1}{3}S(s)x^2$$

sextupole moment

We have

$$x = A\sqrt{\beta}\cos\phi = \sqrt{2J\beta}\cos\phi = \sqrt{2I\beta}\cos\phi$$

$$H = H_0 + \frac{1}{3}S(s)(2\beta I)^{3/2}\cos\phi$$

We expand this in a Fourier series

$$\beta^{3/2}S(s) = \sum_m W_m \cos m\frac{s}{R}$$

$$W_m = \frac{1}{\pi R}\oint \beta^{3/2}S(s)\cos m\frac{s}{R}$$

The rest proceeds as before

Expand the $\cos^3$ terms and just keep the cos terms.

$$H = \frac{\nu}{R} I + \frac{1}{12}(2I)^{3/2} \sum_m W_m \cos\left(m\frac{s}{R}\right)(\cos 3\phi + 3\cos\phi)$$

$$= \frac{\nu}{R} I + \frac{1}{24}(2I)^{3/2} \sum_m W_m \left[\cos\left(3\phi + m\frac{s}{R}\right) + \cos\left(-3\phi + m\frac{s}{R}\right) + 3\cos\left(\phi + m\frac{s}{R}\right) + 3\cos\left(-\phi + m\frac{s}{R}\right)\right]$$

Looking at Hamilton's Equations, we have

$$\frac{dI}{ds} = -\frac{\partial H}{\partial \theta} = -\frac{\partial H}{\partial \phi}$$

$$= \frac{1}{8}(2I)^{3/2} \sum_m W_m \left[-\sin\left(3\phi + m\frac{s}{R}\right) + \sin\left(-3\phi + m\frac{s}{R}\right) - \sin\left(\phi + m\frac{s}{R}\right) + \sin\left(-\phi + m\frac{s}{R}\right)\right]$$

Examine near $3\phi \sim m\frac{s}{R}$

Define a new variable $\tilde{\theta} = \theta - \nu_0 \frac{s}{R}$

$$\longrightarrow \phi = \theta - \nu\frac{s}{R} + \int^s \frac{ds'}{\beta}$$

$$= \tilde{\theta} - \delta\frac{s}{R} + \int^s \frac{ds'}{\beta}; \quad \delta \equiv \nu - \nu_0$$

The part of the Hamiltonian which drives the resonance is

$$H = H_{other} + \frac{\delta}{R} I + \frac{1}{24}(2I)^{3/2} W_m \cos\left((m + 3\delta)\frac{s}{R} - 3\tilde{\theta} - 3\int^s \frac{ds'}{\beta}\right)$$

We now have the equations of motion

$$I' = \frac{dI}{ds} = -\frac{\partial H}{\partial \tilde{\theta}} = \frac{1}{8}(2I)^{3/2} W_m \sin\left((m + 3\delta)\frac{s}{R} - 3\tilde{\theta} - 3\int^s \frac{ds'}{\beta}\right)$$

$$\tilde{\theta}' = \frac{d\tilde{\theta}}{ds} = \frac{\partial H}{\partial I} = \frac{\delta}{R} + \frac{1}{16}(2I)^{1/2} W_m \cos\left((m + 3\delta)\frac{s}{R} - 3\tilde{\theta} - 3\int^s \frac{ds'}{\beta}\right)$$

the fixed points are when the two are zero so

$$\left((m + 3\delta)\frac{s}{R} - 3\tilde{\theta} - 3\int^s \frac{ds'}{\beta}\right) = n\pi$$

The rest proceeds in a similar fashion as before…