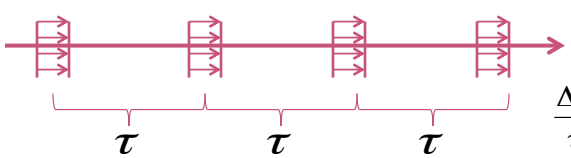
Longitudinal Motion 1

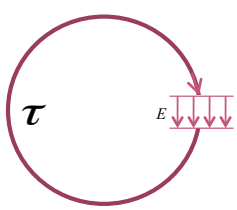
Acceleration in Periodic Structures

- We consider motion of particles either through a linear structure or in a circular ring



Always negative

$$\frac{\Delta\tau}{\tau} = -\frac{1}{\gamma^2}\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$



In both cases, we can adjust the RF phases such that a particle of nominal energy arrives at the the same point in the cycle φ_s

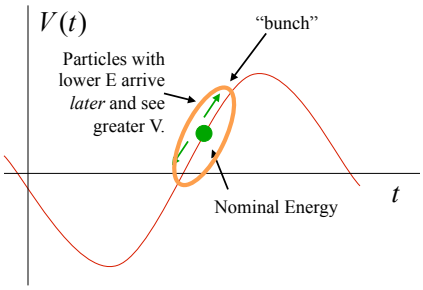
$$\frac{\Delta\tau}{\tau} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} \equiv \eta\frac{\Delta p}{p}$$

Goes from negative to positive at transition

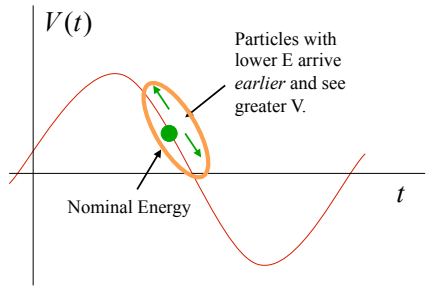
Slip Factors and Phase Stability

- The sign of the slip factor determines the stable region on the RF curve.

η<0 (linacs and below transition)

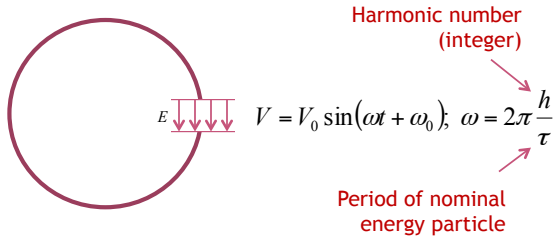


η>0 (above transition)



Longitudinal Acceleration

- Consider a particle circulating around a ring, which passes through a resonant accelerating structure each turn



- The energy gain that a particle of the nominal energy experiences each turn is given by

$$E_{n+1} = E_n + eV_0 \sin\phi_s$$

Synchronous phase

Where the this phase will be the same for a particle on each turn

- A particle with a different energy will have a different phase, which will evolve each turn as

$$\phi_{n+1} = \phi_n + \omega_{rf}\tau\eta\frac{\Delta p}{p} = \phi_n + \frac{\omega_{rf}\tau\eta}{E_S\beta^2}\Delta E$$

use $\frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$

Phase Stability

- Thus the change in energy for this particle for this particle will evolve as

$$\Delta E_{n+1} = \Delta E_n + eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

- So we can write

$$\frac{d\phi}{dn} = \frac{\omega_{rf}\tau\eta}{E_S\beta^2}\Delta E$$

$$\frac{d\Delta E}{dn} = eV_0\left(\sin\phi_n - \sin\phi_s\right)$$

$$\Rightarrow \frac{d^2\phi}{dn^2} = \frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\left(\sin\phi_n - \sin\phi_s\right)$$

- Multiply both sides by $\frac{d\phi}{dn}$ and integrate over *dn*

$$\int \frac{d^2\varphi}{dn^2}\frac{d\varphi}{dn}dn = \frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\int\left(\sin\varphi(n) - \sin\varphi_s\right)\frac{d\varphi}{dn}dn$$

$$\Rightarrow \frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 = -\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\left(\cos\varphi + \varphi\sin\varphi_s\right) + \text{constant}$$

$$\Rightarrow \boxed{\left(\Delta E\right)^2 + 2\frac{eV_0E_S\beta^2}{\omega_{rf}\tau\eta}\left(\cos\varphi + \varphi\sin\varphi_s\right)} = \text{constant}$$

Synchrotron motion and Synchrotron Tune

- Going back to our original equation

$$\frac{d^2\phi}{dn^2} + \left(-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\left(\sin\phi_n - \sin\phi_s\right)\right) = 0$$

- For small oscillations,

$$\sin\phi_n - \sin\phi_s \approx \cos\phi_s\left(\phi_n - \phi_s\right) = \Delta\phi\cos\phi_s$$

- And we have

$$\frac{d^2\Delta\phi}{dn^2} + \left(-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s\right)\Delta\phi = 0$$

- This is the equation of a harmonic oscillator with

$$\omega_n = \sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s} \Rightarrow \nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

Angular frequency wrt *turn* (not time)

"synchrotron tune" = number of oscillations per turn (usually <<1)

Longitudinal Emittance

- We want to write things in terms of time and energy. We have can write the longitudinal equations of motion as

$$\Delta t(n) = \frac{1}{\omega_{rf}} \Delta\phi(n)$$

$$\frac{d\Delta t(n)}{dn} = \frac{1}{\omega_{rf}} \frac{d\Delta\phi(n)}{dn} = \frac{\tau\eta}{E_S\beta^2} \Delta E(n)$$

- We can write our general equation of motion for out of time particles as

$$\Delta t(n) = \Delta t_0 \cos(2\pi\nu_s n) + \frac{1}{2\pi\nu_s} \left.\frac{d\Delta t}{dn}\right|_0 \sin(2\pi\nu_s n)$$

$$= \Delta t_0 \cos(2\pi\nu_s n) + \frac{\tau\eta}{2\pi E_S\beta^2\nu_s} \Delta E_0 \sin(2\pi\nu_s n)$$

$$\Delta E(n) = \frac{E_S\beta^2}{\tau\eta} \frac{d\Delta t(n)}{dn}$$

$$= \Delta E_0 \cos(2\pi\nu_s n) - \frac{2\pi E_S\beta^2\nu_s}{\tau\eta} \Delta t_0 \sin(2\pi\nu_s n)$$

- So we can write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_s n) & \frac{\tau\eta}{2\pi E_S\beta^2\nu_s}\sin(2\pi\nu_s n) \\ -\frac{2\pi E_S\beta^2\nu_s}{\tau\eta}\sin(2\pi\nu_s n) & \cos(2\pi\nu_s n) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- We see that this is the same form as our equation for longitudinal motion with α=0, so we immediately write

$$\begin{pmatrix} \Delta t(n) \\ \Delta E(n) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu_s n) & \beta_L \sin(2\pi\nu_s n) \\ -\gamma_L \sin(2\pi\nu_s n) & \cos(2\pi\nu_s n) \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta E_0 \end{pmatrix}$$

- Where

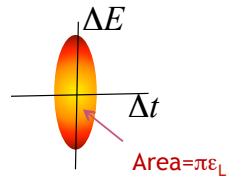
$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \frac{\tau|\eta|}{2\pi E_S\beta^2\nu_s} = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}}; \gamma_L = \frac{1}{\beta_L}$$

- We can define an invariant of the motion as

$$\frac{1}{\beta_L}(\Delta t)^2 + \beta_L(\Delta E)^2 \equiv \epsilon_L = \text{constant}$$

units
generally eV-s



- What about the behavior of Δt and ΔE separately?

$$\Delta E_{RMS} = \sqrt{\frac{\epsilon_L}{\beta_L}} = \left(-\frac{\epsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\varphi_s}{\tau\eta}\right)^{\frac{1}{4}}$$

$$\Delta t_{RMS} = \sqrt{\epsilon_L \beta_L} = \left(-\frac{\tau\eta}{\epsilon_L^2 e V_0 \omega_{rf} E_S \beta^2 \cos\varphi_s}\right)^{\frac{1}{4}}$$

- Note that for linacs or well-below transition

$$\eta = -\frac{1}{\gamma^2} \Rightarrow \Delta E \propto \left(\gamma^3\beta^2\right)^{\frac{1}{4}}; \Delta t \propto \left(\gamma^3\beta^2\right)^{-\frac{1}{4}}$$

Large Amplitude Oscillations

- We can express period of off-energy particles as

$$\frac{dt_n}{dn} = \tau\eta\frac{\Delta p}{p} = \frac{\tau\eta}{E_s\beta^2}\Delta E_n$$

$$\frac{d}{dn}(\Delta E_n) = eV_0\left[\sin(\omega_{rf}t_n) - \sin\phi_s\right]$$

- So

$$\frac{d}{d\phi}(\Delta E) = \frac{d}{d\phi}(\Delta E)\frac{dn}{dt}\frac{dt}{d\phi}$$

$$= \frac{\frac{d}{d\phi}(\Delta E)}{\frac{dt}{dn}\frac{d\phi}{dt}} = \frac{\frac{d}{d\phi}(\Delta E)}{\frac{dt}{dn}\omega_{rf}}$$

$$= \left[\sin\phi - \sin\phi_s\right]\frac{eV_0E_s\beta^2}{\omega_{rf}\tau\eta\Delta E}$$

$$= -\left[\sin\phi - \sin\phi_s\right]\frac{1}{\omega_{rf}^2\cos\phi_s\beta_L^2\Delta E}$$

Use:

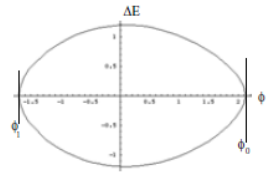
$$\beta_L^2 = -\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}$$

- Continuing

$$\Delta E d(\Delta E) = -\frac{[\sin\phi - \sin\phi_s]}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$

- Integrate

$$\frac{1}{2}(\Delta E)^2 = \frac{[\cos\phi + \phi\sin\phi_s] - [\cos\phi_0 + \phi_0 \sin\phi_s]}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$

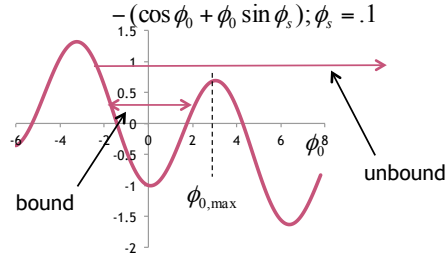


- The curve will cross the ϕ axis when $\Delta E=0$, which happens at two points defined by

$$\cos\phi_1 + \phi_1 \sin\phi_s = \cos\phi_0 + \phi_0 \sin\phi_s$$

- Phase trajectories are possible up to a maximum value of ϕ_0. Consider

$$-(\cos\phi_0 + \phi_0 \sin\phi_s); \phi_s = .1$$



Limit is at maximum of

$$\sin\phi_{0,\max} - \sin\phi_s = 0$$

$$\Rightarrow \phi_{0,\max} = \phi_s \text{ or } \boxed{\pi - \phi_s}$$

Longitudinal Separatrix

- The other bound of motion can be found by

$$\cos\phi_{1,\max} + \phi_{1,\max}\sin\phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s$$
$$= -\cos\phi_s + (\pi - \phi_s)\sin\phi_s$$

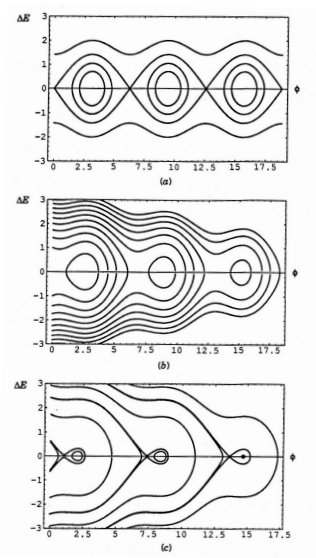
- The limiting boundary (separatrix) is defined by

$$(\Delta E)^2 = 2\frac{[\cos\phi + \phi\sin\phi_s] + [\cos\phi_s - (\pi - \phi_s)\sin\phi_s]}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$

- The maximum energy of the "bucket" can be found by setting $\phi=\phi_s$

$$(\Delta E_b)^2 = 2\frac{2\cos\phi_s + 2\phi_s \sin\phi_s - \pi\sin\phi_s}{\omega_{rf}^2 \cos\phi_s \beta_L^2}$$

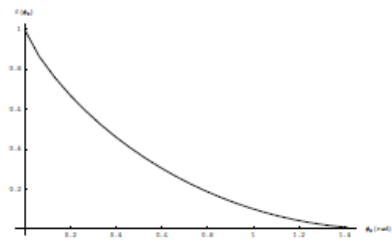
$$\Delta E_b = 2\frac{\sqrt{1 - \left(\frac{\pi}{2} - \phi_s\right)\tan\phi_s}}{\omega_{rf}\beta_L}$$



Bucket Area

- The bucket area can be found by integrating over the area inside the separatrix (which I won't do)

$$A_b = \frac{16\beta}{\omega_{rf}}\sqrt{\frac{eV\cdot E_s}{2\pi h|\eta|}}f(\varphi_s);\quad f(\varphi_s) =$$



Transition Crossing

- We learned that for a simple FODO lattice $\gamma_T \approx \nu$ so electron machines are always above transition.
- Proton machines are often designed to accelerate through transition.
- As we go through transition $(\eta < 0) \Rightarrow (\eta = 0) \Rightarrow (\eta > 0)$
- Recall

$$\nu_s = \frac{1}{2\pi}\sqrt{-\frac{eV_0\omega_{rf}\tau\eta}{E_S\beta^2}\cos\phi_s}$$

$$\beta_L = \sqrt{-\frac{\tau\eta}{eV_0\omega_{rf}E_S\beta^2\cos\phi_s}} = \frac{\Delta t_{\max}}{\Delta E_{\max}}$$

At transition:

$\Delta t_{\max} \Rightarrow \text{constant}$

$\Delta E_{\max} \Rightarrow \infty$

so these both go to zero at transition.

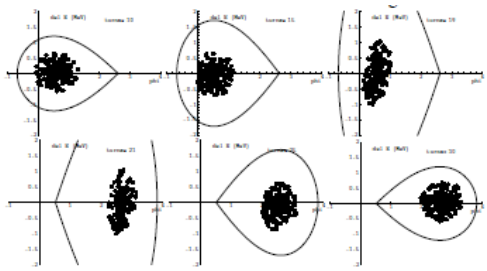
- To keep motion stable

$$\cos\phi_s > 0 \text{ below transition;} \Rightarrow 0 < \phi_s < \frac{\pi}{2}$$

$$\cos\phi_s < 0 \text{ above transition;} \Rightarrow \frac{\pi}{2} < \phi_s < \pi$$

Effects at Transition

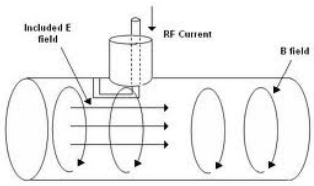
- As the beam goes through transition, the stable phase must change



- Problems at transition (pretty thorough treatment in S&E 2.2.3)
  - Beam loss at high dispersion points
  - Emittance growth due to non-linear effects
  - Increased sensitivity to instablities
  - Complicated RF manipulations near transition
    - Much harder before digital electronics

Accelerating Structures

- The basic resonant structure is the "pillbox"



$$\vec{E} = E(r,t)\hat{z}$$

$$\vec{B} = B(r,t)\hat{\phi}$$

Maxwell's Equations Become:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial E_z}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \frac{\partial E_z}{\partial r} = \frac{\partial B_\theta}{\partial t}$$

Boundary Conditions:

$$\boxed{E_\parallel = B_\perp = 0}$$

Differentiating the first by δt and the second by δr:

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right) = \frac{\partial B_\theta}{\partial r\partial t} + \frac{1}{r}\frac{\partial B_\theta}{\partial t} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial r^2} = \frac{\partial B_\theta}{\partial r\partial t}$$

$$\Rightarrow \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

- General solution of the form

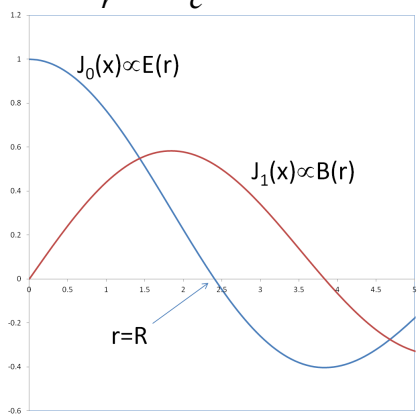
$$E_z = E(r)e^{+i\omega t}$$

- Which gives us the equation

0th order Bessel's Equation

$$E_z'' + \frac{1}{r}E_z' + \frac{\omega^2}{c^2}E_z = 0 \Rightarrow E(r) = E_0 J_0\left(\frac{\omega}{c}r\right)$$

0th order Bessel function



First zero at *J(2.405)*, so lowest mode

$$f_0 = 2.405\frac{c}{2\pi R}$$

$$\frac{\partial E_z}{\partial r} = E_0\frac{\omega}{c}J_0'\left(\frac{\omega}{c}r\right)e^{i\omega t} = \frac{\partial B_\theta}{\partial t} = i\omega B(r)e^{i\omega t}$$

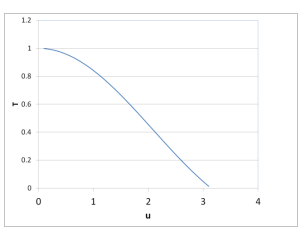
$$\Rightarrow B(r) = -i\frac{1}{c}E_0 J_0'\left(\frac{\omega}{c}r\right) = i\frac{1}{c}E_0 J_1\left(\frac{\omega}{c}r\right)$$

Transit Factor

- In the lowest pillbox mode, the field is uniform along the length (v_p=∞), so it will be changing with time as the particle is transiting, thus a very long pillbox would have no net acceleration at all. We calculate a "transit factor"

Assume peak in middle

$$T \equiv \frac{\text{energy gain}}{eE_0L} = \frac{eE_0\int_{-L/2}^{L/2}\cos\left(2\pi f\frac{z}{v}\right)dz}{eE_0L} = \frac{\frac{v}{\pi f}eE_0\sin\left(\frac{\pi fL}{v}\right)}{eE_0L} = \frac{\sin\left(\frac{\pi fL}{v}\right)}{\left(\frac{\pi fL}{v}\right)} \equiv \frac{\sin u}{u}$$



Example:
- 5 MeV Protons (*v~.1c*)
- *f*=200MHz
- *T*=85%➔*u~1*

$$R = 2.405\frac{c}{2\pi f} = 57\text{ cm}$$

$$L = u\frac{v}{\pi f} = 4.9\text{ cm}$$

Sounds kind of short, but is that an issue?

Power dissipation in RF Cavities

- Energy stored in cavity

Volume=$L\pi R^2$

$$U = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dV = \frac{1}{2}\varepsilon_0\int E_{\max}dV = \frac{1}{2}\varepsilon_0 L E_0^2\int_0^R 2\pi r J_0^2\left(\frac{2\pi f}{c}r\right)dr = \boxed{\frac{1}{2}\varepsilon_0 V E_0^2 J_1^2(2.405)}$$

=$(.52)^2$~25%

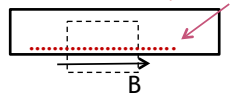
- Power loss:

Magnetic field at boundary

Surface current density J [A/m]

$$\oint \vec{B}\bullet d\vec{l} = LB_\theta = \mu_0 I_{enclosed} = \mu_0 LJ$$

$$\Rightarrow J = \frac{1}{\mu_0}B_\theta$$



Average power loss per unit area is

$$\langle p\rangle = \langle \rho_s J^2\rangle = \frac{1}{2}\frac{1}{\mu_0^2}\rho_s|B_\theta|^2$$

Average over cycle

Cylinder surface

2 ends

$$P = \frac{1}{2}\rho_s\left(\frac{E_0}{\mu_0 c}\right)^2\left(2\pi RLJ_1^2\left(\frac{\omega}{c}R\right) + 2\times 2\pi\int_0^R J_1^2\left(\frac{\omega}{c}r\right)rdr\right)$$

$$\boxed{= \frac{1}{2}\rho_s\left(\frac{E_0}{Z_0}\right)^2 2\pi RL\left(1+\frac{R}{L}\right)J_1^2(2.405)}$$

where $Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c = \frac{1}{\varepsilon_0 c} = 376.73\,\Omega$ (impedance of free space)

- The figure of merit for cavities is the Q, where

$$Q \equiv 2\pi\frac{(\text{Stored Energy})}{(\text{Energy Lost per Cycle})} = \omega\frac{U}{P}$$

$$= \omega\frac{\varepsilon_0 E_0^2\pi R^2 LJ_1^2(2.405)}{\rho_s\left(\frac{E_0}{Z_0}\right)^2 2\pi RL\left(1+\frac{R}{L}\right)J_1^2(2.405)}$$

$$c\varepsilon_0 = \frac{1}{Z_0}$$

$$= \frac{Z_0^2\omega R\varepsilon_0}{2\rho_s\left(1+\frac{R}{L}\right)} = 2.405\frac{Z_0^2 c\varepsilon_0}{2\rho_s\left(1+\frac{R}{L}\right)}$$

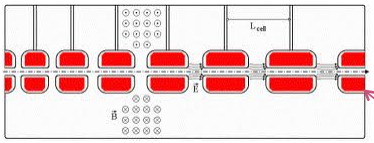
$$= 2.405\frac{Z_0}{2\rho_s\left(1+\frac{R}{L}\right)}$$

- So Q not very good for short, fat cavities!

Drift Tube (Alvarez) Cavity

- Put conducting tubes in a larger pillbox, such that inside the tubes *E=0*

Bunch of pillboxes



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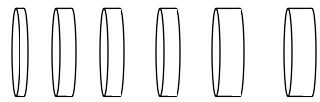
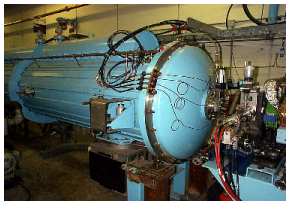


Drift tubes contain quadrupoles to keep beam focused

$$d = \frac{v}{f}$$

Gap spacing changes as velocity increases



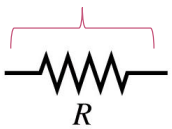
Fermilab low energy linac

➔

Inside

Shunt Impedance

- If we think of a cavity as resistor in an electric circuit, then



$$P = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{P}$$

- By analogy, we define the "shunt impedance" for a cavity as

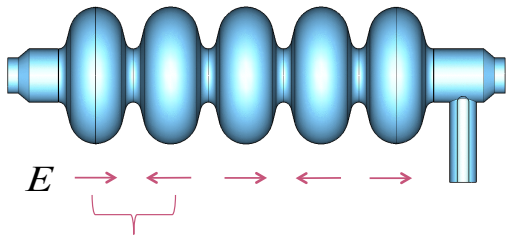
$$R_s \equiv \frac{(\text{voltage gain})^2}{P} = \frac{(E_0 L T)^2}{P}$$

$$= \frac{Z_0^2}{\pi \rho_s} \frac{L}{R} \frac{T^2}{\left(1 + \frac{R}{L}\right) J_1^2(2.405)}$$

We want R_s to be as large as possible

Other Types of Accelerating Structures

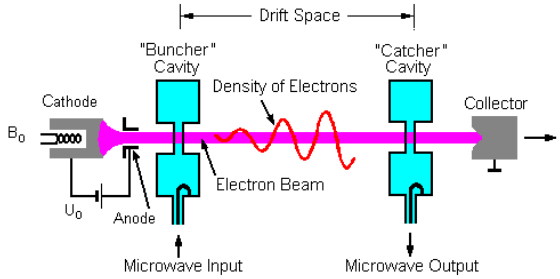
- π cavities



$$d = \pi \frac{v}{\omega}$$

Sources of RF Power

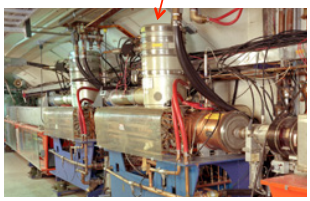
- For frequencies above ~300 MHz, the most common power source is the "klystron", which is actually a little accelerator itself



- Electrons are bunched and accelerated, then their kinetic energy is extracted as microwave power.

Sources of RF Power (cont'd)

- For lower frequencies (<300 MHz), the only sources significant power are triode tubes, which haven't changed much in decades.

FNAL linac 200 MHz Power Amplifier

53 MHz Power Amplifier for Booster RF cavity