Lorentz Transformations (Appendix A)

four-vector of coordin

If you have a particle with coordinates (x, y, z, ct) in an inertial frame S, then its coordinates in a second inertial frame S' moving with uniform velocity v (or β) relative to S is given by Lorentz transformation,

$$x' = \gamma(x - \beta t)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

~~Similarly~~ And the length of this 4-vector is invariant, ie., $c^2t^2 - x^2 - y^2 - z^2 = ct'^2 - x'^2 - y'^2 - z'^2 = ds^2$

$= \text{Constant}$.

Similarly, ~~the~~ Energy and momentum of a particle forms a <u>four-vector</u> (p_x, p_y, p_z, E) and has similar Lorentz transformation and invariance.

$$p_x' = \gamma(p_x - \beta E), \quad p_y' = p_y, \quad p_z' = p_z$$

and $E' = \gamma(E - \beta p_x)$

and $E^2 - p_x^2 - p_y^2 - p_z^2 = E'^2 - p_x'^2 - p_y'^2 - p_z'^2 = m^2$

is invariant.

This can also be written as,

$$\begin{pmatrix} E' \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Note: I, inadvertant switched x and z axes. The frames are in relative motion along z here, instead of x.

Since $p_x' = p_x$; $p_y' = p_y$

This can be compactified as

$$\begin{pmatrix} E' \\ p_\ell' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_\ell \end{pmatrix} \quad \text{and} \quad p_T = p_T'$$

where $p_\ell = p_z$ (the direction of relative motion between the two systems)

$\vec{p}_T = \hat{a}\, p_x + \hat{b}\, p_y$ has the transverse component (i.e., perpendicular to the direction of relative motion.)

The Four-vector Notation

It is useful to use short-hand notations, both for elegance and ease.

4-vector for space-time coordinates, in contravariant notation

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) \equiv (x^0, \vec{x})$$

The covariant notation would be,

$$x_\mu \equiv (ct, -x, -y, -z)$$

The two are connected by

$$x_\mu = g_{\mu\nu} x^\nu \qquad \text{where} \qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Minkowski Metric
or Minkowski tensor

The invariant length of the 4-vector is

$$x^2 = |x|^2 = g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - |\vec{x}|^2$$
$$= c^2t^2 - x^2 - y^2 - z^2$$

In general, $a_\mu = g_{\mu\nu} a^\nu$; $a^\mu = g^{\mu\nu} a_\nu$

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$
$$= a^0 b^0 - \vec{a} \cdot \vec{b}$$

superscripts are indices

$$\therefore a^2 = (a^0)^2 - |a|^2$$

The Energy-momentum four-vector

Also called, **Four-momentum**

$$p^\mu \equiv (E, \vec{p}) = (E, p_x, p_y, p_z) \quad \text{Contravariant}$$

$$p_\mu = g_{\mu\nu} p^\mu = (E, -\vec{p}) = (E, -p_x, -p_y, -p_z)$$

Covariant

$$p^\mu p_\mu = E^2 - p^2 = m^2 \quad \text{Invariant mass}$$

$$x^\mu p_\mu = Et - \vec{p}\cdot\vec{r} \quad \text{Phase}$$

Four-vectors written as p^μ or p

Four-vector scalar product: $p^\mu q_\mu$ or $p\cdot q$

Three-vectors: $\vec{p}$

Quantities in CM frame: $\vec{p}^*$, p^* etc.

We have come across

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad ; \quad \Box = \partial_\mu \partial^\mu$$

Center of Mass Energy and Momentum

In the collision of two particles of masses m_1 and m_2 the total CM energy can be written as

$$W = E_{CM} = \left[(E_1 + E_2)^2 - (p_1 + p_2)^2 \right]^{1/2}$$ Lorentz-invariant

$$= \left[m_1^2 + m_2^2 + 2E_1 E_2 (1 - \beta_1 \beta_2 \cos\theta) \right]^{1/2}$$

where θ is the angle between the two particles.

If we consider the case where one of the particles is at rest, say m_2 is at rest in the lab frame

then $\beta_2 = 0$, $E_2 = m_2$

$$E_{CM} = \left(m_1^2 + m_2^2 + 2 E_{1\,lab}\, m_2 \right)^{1/2}$$

Put $m_1, m_2 \ll E_1$, $m_2 = m_t$, $E_{1\,lab} = E_{beam}$

$\Rightarrow$

$$\boxed{E_{CM} = \sqrt{2 E_{beam} \cdot m_t}}$$

Lab and CM Frames of Reference

Again

Consider a beam of particles with momentum p_L incident on a target at rest in the Laboratory frame. Then the energies of beam particles in the Lab frame is,

$$E_L = \left(m_b^2 c^4 + p_L^2 c^2\right)^{1/2}$$

And the target energy is $E_t = m_t c^2$

The invariant mass W is given by

$$W^2 c^4 = (E_L + m_t c^2)^2 - p_L^2 c^2$$

$$= m_b^2 c^4 + \cancel{p_L^2 c^2} + m_t^2 c^4 + 2 E_L m_t c^2 - \cancel{p_L^2 c^2}$$

$$= m_b^2 c^4 + m_t^2 c^4 + 2 m_t c^2 \cdot E_L$$

or $$E_{CM} = \left(m_b^2 c^4 + m_t^2 c^4 + 2 m_t c^2 \cdot E_L\right)^{1/2}$$

when m_t and m_b are small compared to the beam energy, one can write,

$$E_{CM} = \sqrt{2 m_t c^2 E_L} \quad \text{or} \quad \sqrt{2 m_t E_L}$$

$\because c = 1$.

For example, the pbar production reaction,

$$p + p \rightarrow p + p + p + \bar{p}$$

requires that $E_{CM} \geq 4 m_p$ (in $c = 1$ units)

$$\therefore 4 m_p = \left(2 m_p^2 + 2 m_p E_{min}\right)^{1/2}$$

$$16 m_p^2 = 2 m_p^2 + 2 m_p \cdot E_{min}$$

$$\therefore 2 m_p E_{min} = 14 m_p^2$$

$$\therefore E_{min} = 7 m_p \simeq 6.6 \text{ GeV}/c^2$$

(2) $p + p \rightarrow p + n + \pi^+$

Minimum c.M. energy $= m_p + m_p + m_{\pi^+} = 2.02$ GeV/c

$$\therefore (2.02)^2 = 2 m_p^2 + 2 m_p \cdot E_{min}$$

$$E_{min} = \frac{(2.02)^2 - 2(.938)^2}{2(.938)} = \underline{1.237 \text{ GeV}}$$

$$\boxed{\therefore p_b = 0.80 \text{ GeV}/c}$$

For a system of N (non-interacting) particles, the total energy and momentum become

$$E = E_1 + E_2 + \cdots + E_N \; ; \; p = p_1 + p_2 + \cdots + p_N$$

Then the invariant mass M for a set of N particles can be written as

$$M_{inv}^2 = E^2 - p^2$$

i.e., this has the same value in any reference frame.

In the CM frame of the particles where

$$p = \sum_i p_i = 0$$

$$M_{inv}^2 = E^2 \quad \text{and} \quad M_{inv} = E_{CM}$$

total energy of the CM frame

For a single particle, this reduces to its rest mass.

Masses of Unstable Particles

When a particle decays to N particles

$$A \to 1 + 2 + \cdots + N$$

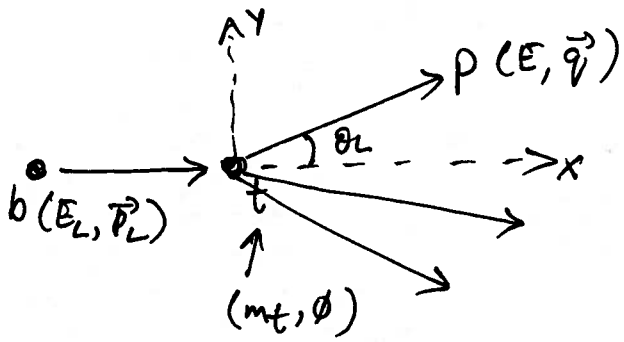
Then the invariant mass of the final state particles

$$M_{inv}^2 = \left(\sum_i E_i\right)^2 - \left(\sum_i p_i\right)^2 = E_A^2 - p_A^2 = m_A^2$$

using Energy-momentum conservation. Mass of the decaying particle A.

Transformation of the Scattering Angle

$$b(E_L, \vec{P}_L) + t(m_t, \emptyset) \rightarrow P(E, \vec{q}) + \cdots$$



Beam direction is along x axis.

Assume the outgoing particle of interest to be in the plane xy.

$$\vec{P}_L = (P_L, 0, 0) \quad , \quad \vec{q} = (q\cos\theta_L, q\sin\theta_L, 0)$$

θ_L = Production or scattering angle

Previously, we worked out velocity of the CM.

$$\beta = \beta_{CM} = \frac{P_L}{E_L + m_t}$$

So, we can calculate the momenta q' of P in CM frame, where θ_c is the angle in CM frame.

$$\vec{q}' = (q'\cos\theta_c, q'\sin\theta_c, 0)$$

$$q'_x = \gamma(q\cos\theta_L - \beta E) \; ; \; q'_y = q\sin\theta_L$$

$$q'_z = 0$$

$$\tan\theta_c = \frac{q_y'}{q_x'} = \frac{1}{\gamma}\,\frac{q\sin\theta_L}{(q\cos\theta_L - \beta E)}$$

For the inverse transformation,

$$\tan\theta_L = \frac{1}{\gamma}\,\frac{q'\sin\theta_c}{q'\cos\theta_c + \beta E'}$$

At high energies, where $E_L \approx p_L \gg m_b, m_t$

$$\tan\theta_L \approx \left(\frac{2m_t}{p_L}\right)^{1/2} \cdot \frac{\sin\theta_c}{\cos\theta_c + 1}$$

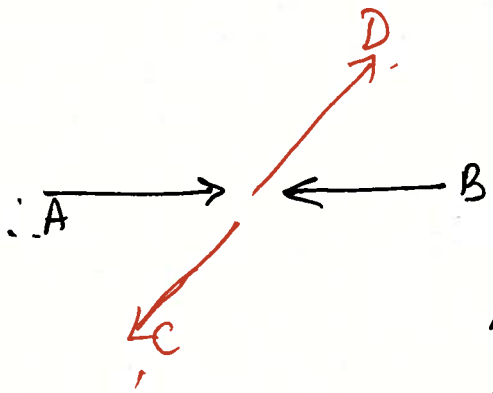
When p_L is very large, θ_L is small.

So, in the lab frame, the particles are produced close to the beam direction, while in the center-of-mass, the particles are more uniformly distributed in 4π.

Mandelstam Variables

Scattering / Annihilation process

$$A + B \rightarrow C + D$$



In 4-vector notation, 4-momentum conservation implies

$$(p_A^\mu + p_B^\mu) = (p_C^\mu + p_D^\mu)$$

One can define 3 kinematic quantities

$$s = (p_A + p_B)^2 = (p_C + p_D)^2$$

$$t = (p_A - p_C)^2 = (p_B - p_D)^2$$

$$u = (p_A - p_D)^2 = (p_B - p_C)^2$$

Note: I have dropped the superscripts but these are 4-vectors

$$s = (p_A + p_B)^2 = (E_A + E_B)^2 - (\vec{p}_A + \vec{p}_B)^2$$

$$= m^2$$

centre-of-mass energy squared

$\sqrt{s}$ is the CM energy.

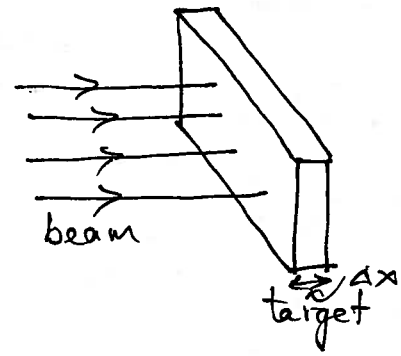
$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

Cross Sections, Amplitudes etc. (M&S Appendix B)

For a reaction

$$b + t \rightarrow c + d + \cdots$$

b: beam, t: target



The flux of beam particles through the target

$$\phi = n_b \cdot v_i$$

Flux = No. of particles per unit area per unit time

where n_b = particle density in beam

v_i = velocity of beam particles relative to target

If "σ_r" is the effective cross section per target particle for a reaction r, then,

number of reactions/unit time

$$W = \phi \cdot \sigma_r \cdot n_t \, dx$$

where n_t = number of particles / unit volume in target (or particle density in target)

dx = thickness of the target.

$L = \phi \cdot n_t \cdot dx$ is referred to as Luminosity. (or equivalent to Luminosity)

The Total Cross Section

"σ_r" defined previously is the cross section for a particular reaction or process.

The total cross section

$$\sigma = \sum_r \sigma_r \qquad \text{or} \qquad \sigma_{tot} = \sum_{i=1}^{n} \sigma_i$$

n possible processes

The total reaction rate

$$W = \sum_r W_r = L\sigma$$

Normally, I write this as

$$N = L\sigma$$

In fact, from previous section,

$$W = \phi \cdot \sigma \cdot n_t \cdot dx$$

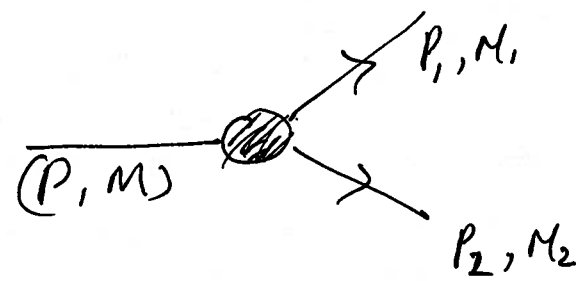
$$= -d\phi \quad \text{(change in flux across } dx\text{)}$$

i.e., $$\frac{d\phi}{\phi} = -n_t \cdot \sigma \cdot dx$$

Integrating, $\phi = \phi_0 \, e^{-x/l_c}$ where $l_c = \dfrac{1}{n_t \cdot \sigma}$

Mean free path or collision length of a particle in a thick target, before it interacts.

Two - Body Decays



In the rest frame of the decaying particle, 4-momentum conservation gives

$$P = P_1 + P_2 \qquad \Big| \text{ or } p^\mu = p_1^\mu + p_2^\mu$$

$$P_1 = P - P_2 \quad , \quad p_1^2 = p^2 + p_2^2 - 2 P P_2$$

using $p^2 = E^2 - m^2$ and $P_1 P_2 = E_1 E_2 - \vec{P}_1 \cdot \vec{P}_2$

we get

$$m_1^2 = M^2 + m_2^2 - 2(E E_1 - \vec{P} \cdot \vec{P}_2)$$

by setting $\vec{P} = 0$ and $E = M$

$$= M^2 + m_2^2 - 2 M E_2$$

$$\therefore \quad E_2 = \frac{M^2 - m_1^2 + m_2^2}{2M}$$

Similarly $\quad E_1 = \dfrac{M^2 - m_2^2 + m_1^2}{2M}$

Transition Rate & Fermi's Golden Rule

The rate for transition from an initial state i to a final state f, is given by Fermi's Golden Rule

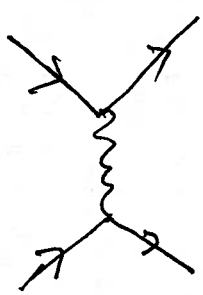
$$W = \frac{2\pi}{\hbar} |M_{if}|^2 \rho_f$$

where M_{if} is the matrix element between initial and final state. M_{if} is also called the Transition matrix element.

$$M_{if} = \int \psi_f^* \hat{H} \psi_i \, dv$$

$\hat{H} \equiv$ Interaction Hamiltonian

It is essentially the overlap integral between the initial and final state brought about by the interaction potential. The matrix element describes the interaction and hence, includes



(1) vertex factors or coupling constants
(2) the propagator terms
(3) energy dependence, angular dependence etc.

$\rho_f = \frac{dN}{dE}$ is the energy density of final states

i.e., the available "phase space" for the final state particles.

For $a+b \to c+d$

Considering particles produced in a solid angle $d\Omega$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} |M_{if}|^2 \rho_f \qquad \rho_f = \frac{dN}{dE} = \frac{dN}{dp} \cdot \frac{dp}{dE}$$

$$= \frac{2\pi}{\hbar} \frac{|M_{if}|^2}{(2\pi\hbar)^3} \cdot \frac{p_f^2}{v_i v_f}$$

v_i and v_f are the relative velocities of a, b and c, d in the CMS, respectively.

When we consider spins of particles, the spin-states need to be taken into account by summing over all final state configurations and averaging over initial states.

$$\therefore \frac{d\sigma}{d\Omega}(a+b \to c+d)$$

$$= \frac{1}{4\pi^2\hbar^4} |M_{if}|^2 \cdot \frac{p_f^2}{v_i \cdot v_f} \cdot \frac{(2S_c+1)(2S_d+1)}{(2S_a+1)(2S_b+1)}$$