If we consider a cylindrical shell between b and b+db and in thickness dx, the energy transfered to electrons becomes,

$$-\Delta E(b) = \frac{2Z^2e^4}{mb^2v^2} \cdot n_e \cdot 2\pi b \cdot db \cdot dx$$

$$-\frac{dE}{dx} = \frac{4\pi n_e Z^2 e^4}{mv^2} \int_{b_{min}}^{b_{max}} \frac{db}{b} = \frac{4\pi Z^2 e^4}{mv^2} \ln \frac{b_{max}}{b_{min}} \quad -(3)$$

Use physical arguments to estimate b_{min}, b_{max}

The maximum transferable energy in a head-on collision is $(2\gamma m_e v)^2 / 2m_e = 2\gamma^2 m v^2$

$$2\gamma^2 m v^2 = \frac{2Z^2e^4}{m b_{min}^2 v^2} \quad \text{using (2)} \quad \therefore b_{min} = \frac{Ze^2}{\gamma m v^2}$$

The interaction must take place in a time short compared to the orbital period of the atomic electrons

$$t_{int} = \frac{b}{\gamma v} = \frac{1}{\nu} \quad \therefore b_{max} = \frac{\gamma v}{\nu}$$

$$\boxed{-\frac{dE}{dx} = \frac{4\pi n_e z^2 e^4}{m v^2} \ln \frac{\gamma^2 m v^3}{z e^2 \nu}}$$

Bohr's Classical Formula

A semiclassical approach: for b_{min}, b_{max}:

Assume min. impact parameter to be half the de Broglie wavelength of the electron

$$b_{min} = \frac{h}{2p} = \frac{h}{2\gamma m v}$$

As before,

$$b_{max} = \frac{\gamma v}{\nu} = \frac{h \cdot \gamma v}{h\nu} = \frac{h \cdot \gamma v}{I}$$

I = Ionization potential

$$\boxed{-\frac{dE}{dx} = \frac{4\pi n_e z^2 e^4}{m v^2} \ln \frac{2\gamma^2 m v^2}{I}}$$

Quantum Treatment:

A proper treatment of energy loss must take into account (1) that energy transfers to atomic electrons occur in discrete amounts and (2) the wave nature of particles. Such a treatment was done by Bethe & Bloch in early 1930's.

$$-\frac{dE}{dx} = n_e \int_0^\infty W \frac{d\sigma}{dW}(E, W)\, dW$$

$\frac{d\sigma}{dW}(E, W)$ = cross section for an incident particle with energy E to lose an amount of energy W in the collision with an electron. The transition probabilities for excitation or ionization of an electron are calculated using first order perturbation theory. The incident particle is treated as a plane wave.

$$-\frac{dE}{dx} = \frac{4\pi n_e Z^2 e^4}{m v^2}\left[\frac{1}{2}\ln \frac{2 m v^2 \gamma^2 E_{kin}^{max}}{I^2} - \beta^2\right]$$

Bethe - Bloch Formula

$$-\frac{dE}{dx} = 4\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A} \cdot \frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2 m_e c^2 \gamma^2 \beta^2 E_{kin}^{max}}{I^2} - \beta^2 - \frac{\delta}{2} - 2\frac{C}{Z}\right]$$

Correction needed at very low energies

N_A = Avogadro Number

r_e = Bohr Radius $= e^2/m_e c^2$

z = charge of the incoming particle

Z = Atomic number of the material

A = Mass number "

δ = Density Effect Parameter

I = Mean Ionization potential $\simeq 16 Z^{0.9}$ eV for $Z > 1$

$\simeq 10 Z$ eV for $Z > 20$

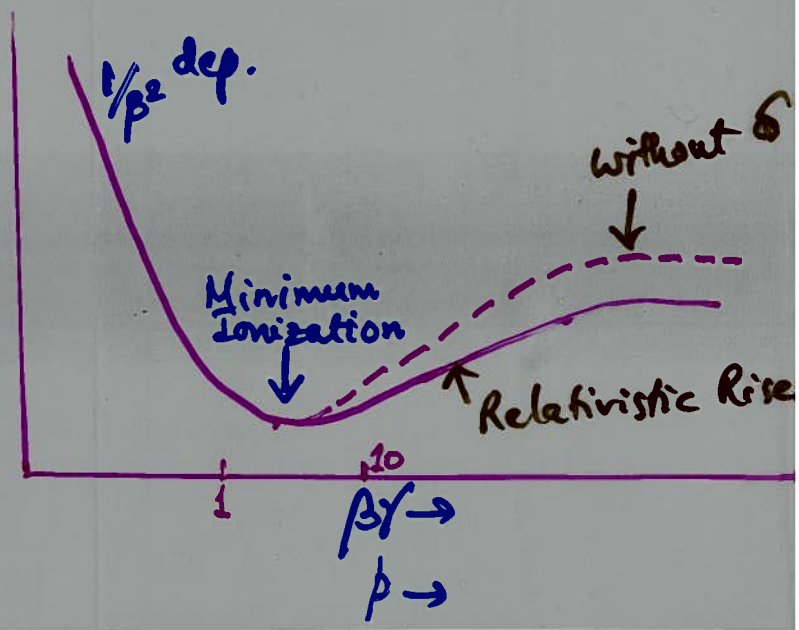
Also, used $n_e = Z \cdot \frac{N_A}{A} \cdot \rho$

and converted dE/dx to energy/gm.cm^{-2}

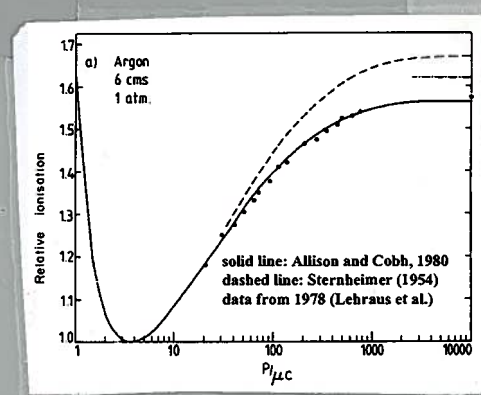
$$K = 4\pi N_A r_e^2 m_e c^2 = .307 \text{ MeV/g}\cdot\text{cm}^{-2}$$

$$-\frac{dE}{dx} = K\, z^2 \cdot \frac{Z}{A} \cdot \frac{1}{\beta^2} \cdot \left[\frac{1}{2} \ln \frac{2 m_e c^2 \gamma^2 \beta^2 E_{kin}^{max}}{I^2} - \beta^2 - \frac{\delta}{2} \right]$$

- $\frac{dE}{dx}$ first falls as $\frac{1}{\beta^2}$
- Reaches minimum at $\beta\gamma \simeq 4$ (MIP)
- Then rises due to $\ln \gamma^2$ term (contributions by more distant collisions)



- Relativistic rise cancelled by "density effect," screening of more distant atoms

$\rightarrow$ Fermi Plateau

$$\left(-\frac{dE}{dx}\right)_{min} \simeq 3.5 \frac{Z}{A}\ \mathrm{MeV/g/cm^2}$$