FERMILAB-PUB-07-160-T
Saclay/SPhT–T07/107

A simple shower and matching algorithm

W. T. Giele[1], D. A. Kosower[2], P. Z. Skands[1]

1 *Theoretical Physics, Fermi National Accelerator Laboratory, MS106, Box 500, Batavia, IL–60510, USA*
2 *Service de Physique Théorique, CEA–Saclay, F–91191 Gif-sur-Yvette cedex, France*

Abstract

We present a simple formalism for parton-shower Markov chains. As a first step towards more complete 'uncertainty bands', we incorporate a comprehensive exploration of the ambiguities inherent in such calculations. To reduce this uncertainty, we then introduce a matching formalism which allows a generated event sample to simultaneously reproduce any infrared safe distribution calculated at leading or next-to-leading order in perturbation theory, up to sub-leading corrections. To enable a more universal definition of perturbative calculations, we also propose a more general definition of the hadronization cutoff. Finally, we present an implementation of some of these ideas for final-state gluon showers, in a code dubbed VINCIA.

1 Introduction

At present collider energies, the strong (QCD) coupling strength α_s is sufficiently large that even the most sophisticated approximations are typically reliable only over a limited region of phase space. Descriptions which work well for "hard" radiation (extra jets, hard bremsstrahlung) break down rapidly in soft and/or collinear regions (jet structure, soft bremsstrahlung), and vice versa. In addition, at scales below a GeV or so, non-perturbative effects must also be taken into account, and a transition made to a description in terms of screened charges inside colorless hadrons. In this paper, we shall put the main focus on aspects which are systematically examined in two complementary perturbative approximations: that of fixed-order truncations, appropriate for "hard" radiation, and that of parton shower resummations, appropriate for "soft/collinear" radiation.

In the past, these two approximations were often pursued independently. The last decade or so has witnessed rapid progress in our understanding of how the virtues of each can be used to overcome the vices of the other, to yield "matched" results which attempt to combine the best features of both approaches. The earliest concrete approach, due to Sjöstrand and collaborators [1], is implemented in the PYTHIA generator [2] and consists of re-weighting the first parton shower emission off a hard system X by a correction factor equal to the ratio needed to reproduce the tree-level $(X+1)$-parton matrix element. This reweighting relies heavily on the shower algorithm(s) in PYTHIA [3] covering all of phase space and on the first shower emission being clearly identifiable as the "hardest". A different approach was needed for the coherent angular-ordered algorithm [4] used in the HERWIG Monte Carlo [5], in which soft gluons can be emitted at large angles "before" harder ones at smaller angles. The approach developed by Seymour for this purpose [6] combines two ingredients: in the region populated by the shower, emissions are reweighted to produce the matrix element rate almost as above (generalized to the angular-ordered case), whereas in the so-called "dead zone," separate $(X+1)$-parton events are generated according to

the appropriate matrix element, and weighted by a Sudakov factor. This technique can be viewed as a precursor to the modern CKKW approach [7].

Although matching beyond one extra parton was attempted within the PYTHIA framework [8], the complexity of the problem grows rapidly. The CKKW and MLM [9] matching schemes broke through this barrier, in principle providing a framework for matching through any number of tree-level matrix elements, though practical applications are still limited to including matrix elements for at most a handful of additional partons. The original CKKW approach is implemented in the SHERPA generator [10]. The MLM one is in principle less dependent on the specific implementation, but is probably most often used with ALPGEN [11] interfaced [12] to either HERWIG or PYTHIA. The basic idea behind the CKKW scheme has since been refined and extended, first via so-called pseudoshowers introduced by Lönnblad [13, 14] in the context of the color dipole model [15, 16] and implemented in the ARIADNE generator [17]; and later by Mrenna and Richardson [18] using MADGRAPH [19], again interfaced to HERWIG and PYTHIA. The most recent advance to be implemented in a widely-used generator was the subtraction-based loop-level matching proposed by Frixione and Webber [20] that led to the MC@NLO [21] add-on to the HERWIG generator.

More recently, several groups have presented proposals to improve MC@NLO-style matching [22]; to include 1-loop contributions in a CKKW-like scheme [23]; to develop a formalism capable of dealing with subleading color and spin effects [24]; to include small-x effects [25]; and to use Soft-Collinear Effective Theory (SCET, see ref. [26] and references therein) as a framework for matched parton showers [27].

Making use of a generalization of subtraction-based matching, our aim is to present a simple formalism for leading-log leading-color (LL) parton showers, constructed explicitly with two main goals in mind: 1) including systematic uncertainty estimates within the LL shower, and 2) combining the virtues of CKKW-type matching (matching to tree-level matrix elements with an arbitrary number of additional partons) with those of MC@NLO-type approaches (matching to 1-loop matrix elements). Moreover, we aim to do this in a manner which is simultaneously simple and does not introduce any dependence on clustering schemes or $p_\perp$ cutoffs beyond those required to regulate explicit subleading logarithms and hadronization effects.

The question of negative weights is generic to subtraction-based approaches, although tricks can be used to circumvent them, as, e.g., in the POWHEG approach [28]. Within the formalism we present, the appearance of negative weights is directly controlled via the choice of subtraction terms. When present, they are associated with corrections that contain no large logarithms. Further, weights of either sign should be devoid of strong infrared peaks since phase space generation only needs to be carried out on matrix elements which 1) have had their leading singularities subtracted out and 2) their subleading ones cut away, as follows.

Though we shall discuss several possible ways of regulating the subleading divergences (subleading in color and/or logs), the simplest is arguably a step function cutoff, e.g. in m^2 or $p_\perp^2$. Similar to the "matching scale" in CKKW approaches, which also serves the purpose of regulating subleading divergences, such a cutoff should be placed at an intermediate scale, between the hard scale and the hadronization scale, in order to prevent logarithms involving $Q_{\rm had}$ from appearing in the final answer. At no point do we claim that our approach is valid beyond leading color, leading log, but it should also be made clear that the only remaining subleading logarithms will involve this intermediate scale, not the hadronization scale.

Finally, as a first step towards making hadronization models (in particular their "tuning") less depen-

dent on the details of the parton shower they are used with, we propose a generalized definition of the hadronization cutoff. Since our perturbative proposal includes infinite families of different shower evolutions, minimizing the differences at the non-perturbative factorization stage is an important ingredient in matching it to hadronization descriptions.

As an explicit proof of concept, we have combined the antenna factorization formalism [29, 30] with that of dipole showers [16] in a code dubbed VINCIA (Virtual Numerical Collider with Interleaved Antennae) [31], which is being developed both stand-alone and as a plug-in to PYTHIA8[1] [32]. Similar to the existing dipole shower implementation, ARIADNE [17], the choice of the antenna-dipole formalism has the added benefit that the shower is built up from coherent color dipoles which reproduce the dipole radiation pattern by construction, and thus there is no need to introduce explicit coherence effects by hand, neither by choice of evolution variable (angular ordering) nor by ad hoc cuts. We thus expect a much milder dependence on the choice of evolution variable than is the case for conventional parton showers.

The organization of the paper is as follows: in section 2, we work out a convenient formalism for general parton showers in which a clear distinction is maintained between each component of the algorithm (evolution variable, phase space, radiation function, and scale choices). This clear distinction, combined with the relative simplicity of the formulation, will play a central role in securing a high degree of analytical control later on. We then expand the shower into partial cross sections of fixed multiplicities of resolved partons. This expansion is used in section 3 to derive a set of matching terms for arbitrary tree- and 1-loop matrix elements, up to corrections of order an infrared cutoff (hadronization scale) and an intermediate scale regulating the subleading logarithms. Section 4 then deals with improving the infrared factorization between parton showers and hadronization models. The numerical implementation of many of our ideas, in the form of the VINCIA code, is then presented in section 5. In section 6, we discuss how one might go further in the perturbative expansion. We then round off with conclusions and outlook in section 7.

2 The Shower Chain

As a starting point, consider a Markov chain algorithm [33] ordered in some measure of time t. Such chains characterize the development of a broad variety of systems. In our application, the system will be a set of partons and the rôle of t will be played by a measure of parton resolution scale $Q = 1/t$, but the chain could equally well represent the real-time evolution of a simple system such as an ensemble of radioactive nuclei. Two global quantities characterize such evolution: the starting configuration and the duration of the experiment $t_{\rm end} - t_{\rm in}$. At each differential time step in between, there is a probability density $A(t)$ for the system to undergo a non-trivial change. After such a change, $A(t)$ itself may change (for example, a nucleus may be replaced by its decay products). For the chain to have the Markov property, all that is required is that $A(t)$ depend only on the system's present state, not on its past history. This property will turn out to be useful in the context of higher-order matching in section 3.

Let us write the Sudakov factor [34], the probability that the system does not change state between

[1] Many thanks to T. Sjöstrand for making this possible.

two times $t_{\rm in}$ and $t_{\rm end}$, as:

$$\Delta(t_{\rm in}, t_{\rm end}) = \exp\left(-\int_{t_{\rm in}}^{t_{\rm end}} {\rm d}t\ A(t)\right)\ , \tag{1}$$

with the understanding that A can depend on the particular system configuration at time $t_{\rm in}$, and thereby that Δ implicitly has such a dependence as well.

In the parton shower context, $A(t)$ is the total parton evolution or splitting probability density. This includes sums and integrals over all possible types of transitions, such as gluon-to-gluon pair or gluon-to-quark pair splitting. As our aim is to resum the leading singularities, $A(t)$ must necessarily be infrared divergent: there is an infinite probability to radiate a soft or collinear gluon. The evolution variable must therefore itself be infrared safe, such that all the singularities of A correspond to "late times" as defined by t (or, equivalently, $1/Q$). Specifically, for leading-log evolution t must regulate at least all leading-log divergences.

Letting $\{p\}_n$ denote a complete specification of an n-parton configuration in the leading-color limit (carrying not only information on momenta, but also the color ordering, flavors, and perhaps polarizations), we define the leading-log Sudakov factor by,

$$\Delta(t_{\rm n}, t_{\rm end}; \{p\}_n) = \exp\left(-\int_{t_{\rm n}}^{t_{\rm end}} {\rm d}t_{n+1} \sum_{i\in\{n\to n+1\}} \int \frac{{\rm d}\Phi_{n+1}^{[i]}}{{\rm d}\Phi_n} \delta(t_{n+1} - t^{[i]}(\{p\}_{n+1})) A_i(\{p\}_n \to \{p\}_{n+1})\right)\ , \tag{2}$$

where ${\rm d}\Phi_n$ denotes the n-particle Lorentz-invariant phase-space measure, so that ${\rm d}\Phi_{n+1}^{[i]}/{\rm d}\Phi_n$ represents the branching phase space, and the t-ordering is imposed via the integral over t_{n+1} together with the δ function. This definition will be the cornerstone for the remainder of this paper, and so we now devote a few paragraphs to its explanation. To simplify the notation, we shall usually let the dependence on $\{p\}_n$ be implicit, letting $\Delta(t_{\rm n}, t_{\rm end}) \equiv \Delta(t_{\rm n}, t_{\rm end}; \{p\}_n)$.

The first important aspect is that we define the Sudakov factor not for a lone parton or color dipole, but rather for the n-parton configuration as a whole. The degree to which the evolution of smaller subsystems factorize will naturally play an important rôle but does not need to be specified explicitly at this point. The sum over $i \in \{n \to n+1\}$ runs over all possible ways of obtaining $n+1$ partons from the original n ones. For example, starting with an n-parton configuration of which n_q partons are quarks, an unpolarized parton shower with four active quark flavors would yield one term for each quark in the event ($n_q \times q \to qg$) and five terms for each gluon $(n - n_q) \times (n_f \times g \to q\bar{q} + g \to gg)$.

The evolution phase space is represented symbolically by the $(n+1)$-parton phase space for an evolution step of type i, ${\rm d}\Phi_{n+1}^{[i]}$, divided by that of the evolving configuration, ${\rm d}\Phi_n$. Its specification requires three variables, along with a mapping from these variables to the phase space for the emission. Existing parton-shower Monte Carlo implementations each choose a different function for this map. To name a few known issues, the map may have 'dead zones' where it is zero, and/or it may have regions where several independent emitters i populate the same point (such double counting is not necessarily a problem, so long as the sum is properly normalized); in some formulations, the entire event may participate in each branching, in others only a specific pair of partons 'recoil' off each other; in analytical leading-log (LL) resummations, a purely collinear map is usually used, which slightly violates momentum conservation, and so on. Our point is not which choice is 'best', but that many are possible, each leading to a different shower evolution.

In this paper, we shall require that the partons at each step be on shell and that four-momentum be

conserved. An $(n+1)$-parton phase space then has $3(n+1)-4$ degrees of freedom:

$$\mathrm{d}\Phi_{n+1} = (2\pi)^4 \delta^{(4)} \left(\sum_{j=1}^{n+1} p_j - \sum_{j=1}^{n} \hat{p}_j \right) \prod_{j=1}^{n+1} \frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3 2E_j} \ , \tag{3}$$

where we denote the n original momenta by $\hat{p}$, and the notation for four-vectors is $p_i = (E_i, \mathbf{p}_i)$. This represents all possible $(n+1)$-parton configurations consistent with energy and momentum conservation. In the context of evolution, however, we are already implicitly integrating over all possible n-parton configurations, and we are also explicitly summing over all possible evolution possibilities i for each such configuration. The notation $\mathrm{d}\Phi_{n+1}^{[i]}/\mathrm{d}\Phi_n$ is thus intended to signify the subdivision of the full phase space into discrete (but possibly overlapping) regions, each corresponding to a specific n-parton configuration and a specific i. Since the n-parton phase space has $3n-4$ degrees of freedom, this amounts to imposing $3n-4$ additional δ functions and adding the explicit superscript $[i]$:

$$\frac{\mathrm{d}\Phi_{n+1}^{[i]}}{\mathrm{d}\Phi_n} = \delta^{(4)} \left(\sum_{j=1}^{n+1} p_j - \sum_{j=1}^{n} \hat{p}_j \right) \prod_{j=1}^{n+1} \frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3 2E_j} \delta^{(3n-4)} \left(\{\kappa_i^{-1}(\{p\}_{n+1})\}_n - \{\hat{p}\}_n \right) \ , \tag{4}$$

where κ_i^{-1} is the inverse of the map discussed above. This inverse can be viewed as a clustering definition[2] that, given i and the $(n+1)$-parton configuration, reconstructs the corresponding "unevolved" one, $\{\hat{p}\}_n$. The requirements on the map κ_i are thus:

1. For each i, a unique inverse of κ_i must exist (κ_i must be injective), however each individual κ_i does not necessarily have to cover all of phase space, since we only care about the coverage after summing over i and integrating over $\mathrm{d}\Phi_n$.

2. After summing over i and integrating over $\mathrm{d}\Phi_n$, the resulting composite map should cover all of phase space (be surjective), in order to avoid creating dead zones. It does not necessarily have to be one-to-one: the $(n+1)$-parton phase space may be covered several times so long as this is properly taken into account in the normalization of the radiation functions A_i (or more precisely, of their singular parts).

Obviously, these statements only apply to configurations that are supposed to be obtainable via shower branchings in the first place, and not, for instance, to subleading color topologies such as $Z \to \bar{q}qgg$ with the two gluons in a color singlet state.

Below, we shall restrict our attention to maps corresponding to dipole-antenna showers, such that:

$$\text{Dipole Showers:} \quad \frac{\mathrm{d}\Phi_{n+1}^{[i]}}{\mathrm{d}\Phi_n} = \frac{\mathrm{d}\Phi_{n-2}^{[i]}}{\mathrm{d}\Phi_{n-2}} \frac{\mathrm{d}\Phi_3^{[i]}}{\mathrm{d}\Phi_2} \ . \tag{5}$$

The first factor on the right hand side indicates that we we choose $3(n-2)$ δ functions to leave the momenta of $n-2$ partons unchanged. This leaves us with a dipole-antenna phase space, $\mathrm{d}\Phi_3^{[i]}/\mathrm{d}\Phi_2$, carrying nine degrees of freedom compensated by the six remaining δ functions. Introducing the notation

[2] That is, κ_i^{-1} "inverts" the shower in a manner similar to the action, e.g., of ARCLUS on the ARIADNE shower [17] or PYCLUS on the $p_\perp$-ordered PYTHIA shower.

$\hat{a}+\hat{b}\to a+r+b$ which we shall use for dipole-antenna branchings throughout, four of these delta functions embody overall momentum conservation,

$$\delta^{(4)}(p_a+p_r+p_b-p_{\hat{a}}-p_{\hat{b}}) \quad . \tag{6}$$

The last two delta functions specify the global orientation of the plane spanned by the three daughter partons in the center of mass of the parent dipole (the branching plane), relative to the axis of the parent dipole, in terms of two angles, θ and ψ. Parity conservation fixes one of them so that the branching plane contains the parent dipole axis:

$$\delta(\theta-\hat{\theta}) \quad , \tag{7}$$

where $\hat{\theta}$ is the orientation angle of the parent dipole in a global coordinate system. The δ function in ψ fixes the rotation angle of the daughters around an axis perpendicular to the branching plane. This angle does have a reparametrization ambiguity away from the collinear and soft limits and hence has the following general form

$$\delta(\psi-\hat{\psi}-\psi_{a\hat{a}}) \quad , \tag{8}$$

where $\hat{\psi}$ is the other global orientation angle and the reparametrization term $\psi_{a\hat{a}}$ will be explored further in section 5 (see also ref. [35]). For the time being, we note only that it must vanish in the soft and collinear limits.

The remaining three integration variables we shall map to two invariant masses, $s_{ar}=(p_a+p_r)^2$ and $s_{rb}=(p_r+p_b)^2$, and the last Euler angle, ϕ, describing rotations of the branching plane around the parent dipole axis. The antenna phase space then takes the following form:

$$\frac{\mathrm{d}\Phi_3^{[i]}}{\mathrm{d}\Phi_2} = \frac{\lambda\left(s^{[i]},m_{\hat{a}}^2,m_{\hat{b}}^2\right)^{-\frac{1}{2}}}{16\pi^2}\,\mathrm{d}s_{ar}\mathrm{d}s_{rb}\frac{\mathrm{d}\phi}{2\pi}\,\frac{\mathrm{d}\psi}{\mathrm{d}\hat{\psi}}\frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\hat{\theta}} \tag{9}$$

$$= \frac{1}{16\pi^2 s^{[i]}}\,\mathrm{d}s_{ar}\mathrm{d}s_{rb}\frac{\mathrm{d}\phi}{2\pi} \qquad \text{for} \quad m_{\hat{a}}=m_{\hat{b}}=0 \;\; ; \;\; \psi=\hat{\psi}+\psi_{a\hat{a}} \quad \text{and} \quad \theta=\hat{\theta}\,, \tag{10}$$

where $\lambda(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca$ is the Källén function, $s^{[i]}$ is the invariant mass squared of the branching dipole, and $m_{\hat{a},\hat{b}}$ are the rest masses of the original endpoint partons. The second line represents the massless case, with the two orientation angles θ and ψ fixed as discussed above.

Immediately following the phase space in eq. (2) is a δ function requiring that the integration variable t_{n+1} should be equal to the ordering variable t evaluated on the set of $n+1$ partons, $\{p\}_{n+1}$, i.e. that the configuration after branching indeed corresponds to a resolution scale of t_{n+1}. We leave the possibility open that different mappings will be associated with different functional forms for the post-branching resolution scale, and retain a superscript on $t^{[i]}$ to denote this.

Finally, there are the evolution or showering kernels $A_i(\{p\}_n\to\{p\}_{n+1})$, representing the differential probability of branching, which we take to have the following form,

$$A_i(\{p\}_n\to\{p\}_{n+1}) = 4\pi\alpha_s(\mu_R(\{p\}_{n+1}))\,\mathcal{C}_i\,a_i(\{p\}_n\to\{p\}_{n+1}) \quad , \tag{11}$$

where $4\pi\alpha_s=g_s^2$ is the strong coupling evaluated at a renormalization scale defined by the function μ_R, $\mathcal{C}_i$ is the color factor (e.g. $\mathcal{C}_i=N_c=3$ for $gg\to ggg$), and a_i is a radiation function, giving a leading-logarithmic approximation to the corresponding squared evolution amplitude (that is, a parton or dipole-antenna splitting kernel). When summed over possible overlapping phase-space regions, the

combined result should contain exactly the correct leading soft and collinear logarithms with no over- or under-counting. Non-logarithmic ('finite') terms are in contrast arbitrary. They correspond to moving around inside the leading-logarithmic uncertainty envelope. The renormalization scale μ_R could in principle be a constant (fixed coupling) or running. Again, the point here is not to impose a specific choice but just to ensure that the language is sufficiently general to explore the ambiguity.

Together, eqs. (2), (4), and (11) can be used as a framework for defining more concrete parton showers. An explicit evolution algorithm (whether based on partons, dipoles, or other objects) must specify:

1. The choice of perturbative evolution variable(s) $t^{[i]}$.
2. The choice of phase-space mapping $\mathrm{d}\Phi_{n+1}^{[i]}/\mathrm{d}\Phi_n$.
3. The choice of radiation functions a_i, as a function of the phase-space variables.
4. The choice of renormalization scale function μ_R.
5. Choices of starting and ending scales.

The definitions above are already sufficient to describe how such an algorithm can be matched to fixed order perturbation theory. We shall later present several explicit implementations of these ideas, in the form of the VINCIA code, see section 5.

Let us begin by seeing what contributions the pure parton shower gives at each order in perturbation theory. Since Δ is the probability of no branching between two scales, $1-\Delta$ is the integrated branching probability $\mathcal{P}_{\mathrm{branch}}$. Its rate of change gives the instantaneous branching probability over a differential time step $\mathrm{d}t_{n+1}$:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{P}_{\mathrm{branch}}(t_n,t_{n+1})}{\mathrm{d}t_{n+1}} &= \frac{\mathrm{d}}{\mathrm{d}t_{n+1}}\Big(1-\Delta(t_n,t_{n+1})\Big)\\ &= \sum_i \int \frac{\mathrm{d}\Phi_{n+1}^{[i]}}{\mathrm{d}\Phi_n}\delta(t_{n+1}-t^{[i]}(\{p\}_{n+1}))A_i(\{p\}_n\to\{p\}_{n+1})\Delta(t_n,t_{n+1})\ .\end{aligned} \tag{12}$$

This expression still contains an explicit integral over all phase-space variables except t_{n+1}. The corresponding fully differential distribution of the (time-ordered) branching probability is simply the "naïve" evolution kernel times the Sudakov factor,

$$A_i(\{p\}_n\to\{p\}_{n+1})\Delta(t_n,t_{n+1})\ . \tag{13}$$

We now seek the complete fixed-order expansion for the distribution of an observable $\mathcal{O}$, as computed by the Markov process. By definition, $\mathcal{O}$ is always evaluated on the final configuration, reached once the Markov chain terminates at t_{end}. A convenient physical interpretation of t_{end} is as the hadronization cutoff, t_{had}, beyond which the parton shower is not evolved; since the evolution would receive $\mathcal{O}(1)$ corrections from hadronization beyond this scale, exclusive event properties can only be further probed by switching to a non-perturbative description in that region (alternatively, stopping the shower and evaluating $\mathcal{O}$ on the partons at this scale is still an improvement over fixed-order calculations). We will return to a generalization of this notion in a later section; for now merely take t_{had} as a cutoff in the

evolution variable. This amounts to implicitly treating all radiation (and hadronization effects) below the cutoff inclusively, that is summing over additional partons below that scale.

Starting from any process, X, with differential phase space weight w_X, the parton-shower improved distribution of $\mathcal{O}$ is:

$$\frac{\mathrm{d}\sigma_X}{\mathrm{d}\mathcal{O}}\Big|_{\mathrm{PS}} = \int \mathrm{d}\Phi_X \; w_X \; S(\{p\}_X, \mathcal{O}) \quad . \tag{14}$$

The definition of w_X in the context of matching will be explored in the next section, but for the pure shower w_X is just the tree-level matrix element squared for the parent process X, possibly subject to matrix-element-level cuts or constraints (e.g. Z production restricted to a window around the Z mass, etc). S is a showering operator that generates the Markov chain starting from a list of partons $\{p\}_X$. It is defined by:

$$\begin{aligned} S(\{p\}_X, \mathcal{O}) \;=\; & \underbrace{\delta\left(\mathcal{O} - \mathcal{O}(\{p\}_X)\right) \Delta(t_X, t_{\mathrm{had}})}_{X+0 \text{ exclusive above } 1/t_{\mathrm{had}}} \\ & + \underbrace{\int_{t_X}^{t_{\mathrm{had}}} \mathrm{d}t_{X+1} \sum_i \int \frac{\mathrm{d}\Phi_{X+1}^{[i]}}{\mathrm{d}\Phi_X} \delta(t_{X+1} - t^{[i]}(\{p\}_{X+1})) \Delta(t_X, t_{X+1}) A_i(...) S(\{p\}_{X+1}, \mathcal{O})}_{X+1 \text{ inclusive above } 1/t_{\mathrm{had}}} \;, \end{aligned} \tag{15}$$

where t_X, the starting scale for each successive step of the evolution, depends implicitly on $\{p\}_X$, the integration over t_{X+1} runs over all possible branching scales between t_X and t_{had}, and $A_i(...)$ is defined by eq. (11). Expanding the Markov chain to a few orders will be useful in the context of matching below and simultaneously illustrates explicitly how the chain works:

$$\begin{aligned} & S_X(\{p\}_X, \mathcal{O}) = \\ & \underbrace{\delta\left(\mathcal{O} - \mathcal{O}(\{p\}_X)\right) \Delta(t_X, t_{\mathrm{had}})}_{X+0 \text{ exclusive above } 1/t_{\mathrm{had}}} \\ & + \int_{t_X}^{t_{\mathrm{had}}} \mathrm{d}t_{X+1} \sum_i \int \frac{\mathrm{d}\Phi_{X+1}^{[i]}}{\mathrm{d}\Phi_X} \delta(t_{X+1} - t^{[i]}(\{p\}_{X+1})) \Delta(t_X, t_{X+1}) A_i(...) \\ & \quad \times \Bigg[\underbrace{\Delta(t_{X+1}, t_{\mathrm{had}}) \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+1})\right)}_{X+1 \text{ exclusive above } 1/t_{\mathrm{had}}} \\ & \quad + \int_{t_{X+1}}^{t_{\mathrm{had}}} \mathrm{d}t_{X+2} \sum_j \int \frac{\mathrm{d}\Phi_{X+2}^{[j]}}{\mathrm{d}\Phi_{X+1}^{[i]}} \delta(t_{X+2} - t^{[j]}(\{p\}_{X+2})) \Delta(t_{X+1}, t_{X+2}) A_j(...) \\ & \quad \times \Bigg\{ \underbrace{\Delta(t_{X+2}, t_{\mathrm{had}}) \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+2})\right)}_{X+2 \text{ exclusive above } 1/t_{\mathrm{had}}} + \cdots \Bigg\} \Bigg] \; . \end{aligned} \tag{16}$$

Each underbraced term corresponds to the finite contribution from a specific exclusive final-state multiplicity (exclusive in the 'smeared' sense discussed above, that is, exclusive above scales $Q_{\mathrm{had}} = 1/t_{\mathrm{had}}$

but inclusive for smaller scales and hence still infrared safe). The first underbraced line describes the shower-improved contribution to the distribution of $\mathcal{O}$ from 'events' which have no perturbatively resolvable emissions at all, the second contributions from events which have exactly one resolved emission, and so on.

This expression can now be expanded further, to any fixed order in the coupling. Each A contains one power of α_s, and the exponentials inside Δ must also be expanded. The latter expansion gives rise to higher-order corrections which do not increase the parton multiplicity, and thus correspond to the 'virtual corrections' generated by the shower. The explicit A factors, in contrast, represent the shower approximation of corrections due to real radiation.

3 Matching

There are several possible definitions of what one might mean by 'matching', reflecting the general concept of making two different asymptotic expansions of a given observable agree in an intermediate region; our first task is thus to establish a clear nomenclature. In a perturbative calculation, observables will have expansions in the strong coupling α_s. Each observable will start at some order j in the coupling, and suffer corrections at subsequent orders. As is usual in fixed-order calculations, we will refer to a calculation of the first order for a given observable as being of leading order (LO), a calculation accurate to the following order as next-to-leading order (NLO), and so on. Because of the presence of infrared singularities, a given order in an observable will receive contributions from perturbative amplitudes (matrix elements) of different loop order. We will label perturbative amplitudes of 'bare' partons, complete with their infrared singularities, by loop order[3]. That is, we reserve the nomenclature LO, NLO, etc. for *observables* only, and that of tree-level, 1-loop, etc. for *matrix elements* only.

Event samples as produced by a matched Markov-chain parton shower can be used to measure many different observables. We therefore believe it would be misleading to characterize them as being of leading or next-to-leading order. More properly, they should be characterized in terms of which matrix elements are included in the matching. In this paper, for an arbitrary shower initiator process X (the parent process), we intend that *"X matched to $X+n$ partons at tree level and $X+m$ partons at loop level"* (with $m < n$) should fulfill the following:

- It should resum the leading-color soft and collinear logarithms to all orders.
- For any $j \leq n$, it should reproduce the LO distribution of any observable whose expansion starts at order α_s^j, up to corrections of order α_s^{j+1} and/or $Q_{\rm had}^2/Q_{\rm X}^2$, where $Q_{\rm had}$ is a hadronization scale and Q_X is a hard scale associated with the X process.
- For $k \leq m$, it should also reproduce any such distribution calculated at NLO, that is up to corrections of $\mathcal{O}(\alpha_s^{k+2})$ and $Q_{\rm had}^2/Q_{\rm X}^2$.

The first point corresponds to the pure parton shower, the second to CKKW- or tree-level matching, and the third to a generalized variant of loop-level matching of the sort implemented by MC@NLO. Our purpose here is to combine all three into a unified approach, in which essentially any tree-level or 1-loop matrix element could be incorporated with a minimum of effort.

[3]We will, however, leave the use of and details of dimensional regularization and infrared cancellations implicit.

As an example, consider Z decay. A pure parton shower resums the leading logarithms, but will only be "matched" to $Z+0$ partons at tree level, as it will generically introduce errors of $\mathcal{O}(\alpha_s)$ in any observable. If we match to $Z+1$ parton at tree level, we will correctly reproduce any large-logarithm-free distribution as predicted using the $Z \to \bar{q}gq$ matrix element (the only three-parton leading-order matrix element). If we match to $Z+2$ partons at tree level, we will correctly reproduce four-jet distributions with only $\mathcal{O}(\alpha_s^3)$ corrections, corresponding to the use of tree-level four-parton matrix elements such as $Z \to \bar{q}ggq$. Tree-level matching up to three additional partons combined with 1-loop matching up to two additional partons, will allow us to reproduce four-jet distributions up to corrections of $\mathcal{O}(\alpha_s^4)$, in addition to resumming the leading logarithms, and so forth.

Let us fix an (arbitrary) observable $\mathcal{O}$ as representative of the distribution above, and consider a computation of the cross section, or partial width as the case may be, differentially in $\mathcal{O}$. We seek a prescription that will yield a generated event sample from which distributions can be made that simultaneously fulfill all the three requirements above.

To specify our matching prescription, we introduce two finite matching terms at each order in α_s, one for resolved radiation (R) and one for single unresolved and 1-loop corrections (V). Our expression for the matched-shower-improved (MS) distribution is then:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{O}}\Big|_{\mathrm{MS}} = \sum_{k=0}^{n} \int \mathrm{d}\Phi_{X+k} \left(w_{X+k}^{(R)} + w_{X+k}^{(V)} \right) \Theta(t_{\mathrm{had}} - t(\{p\}_{X+k}) S(\{p\}_{X+k}, \mathcal{O}) \,, \tag{17}$$

where $w_{X+k}^{(R)}$ is the tree-level matching coefficient for $X+k$ partons and $w_{X+k}^{(V)}$ is the corresponding virtual one. Denoting the couplings present in the parent process X collectively by α_X, these matching terms are of order $\alpha_X \alpha_s^k$ and $\alpha_X \alpha_s^{k+1}$, respectively. One or both of them may be zero in the absence of matching. The operator S is the same as above, embodying an all-orders resummation of both real and virtual corrections in the leading-logarithmic approximation.

The tree- and loop-level matching terms $w_{X+k}^{(R)}$ and $w_{X+k}^{(V)}$ may now be derived by expanding the real and virtual terms of S separately and, order by order, comparing the contribution from each fixed parton multiplicity to the observable $\mathcal{O}$ as calculated in a fixed-order expansion:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{O}}\Big|_{\mathrm{ME}} = \sum_{k=0} \int \mathrm{d}\Phi_{X+k} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta(\mathcal{O} - \mathcal{O}(\{p\}_{X+k}) \,, \tag{18}$$

where k still represents the number of legs and ℓ represents the number of loops. It goes without saying that matching at incomplete orders will involve some arbitrariness, which will be explored further below.

3.1 "Matching" to $X+0$ partons at tree level

Matching to "$X+0$ partons at tree level" just means verifying that the lowest-order expansion of the shower is identical to the lowest-order parent matrix element, i.e., that all corrections generated by the shower are of higher order. This is trivially true, but let us verify it explicitly as a first exercise. Only the first line of eq. (16) is relevant, with the Sudakov Δ expanded to unity,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{O}}\Big|_{\mathrm{PS}} \sim \int \mathrm{d}\Phi_X \, w_{X+0}^{(R)} \, \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+0})\right) \,, \tag{19}$$

from which we infer that the trivial condition $w_{X+0}^{(R)} = |M_{X+0}^{(0)}|^2$ ensures that the two descriptions match at lowest order.

3.2 Matching to $X+1$ parton at tree level

At order α_s, the parton shower (PS) generates only one term that contributes to $X+1$ parton, i.e., taking the real radiation term and expanding the Sudakov to unity (cf. the matching definition above):

$$\text{PS} \quad : \quad \int \mathrm{d}\Phi_X \, |M^{(0)}_{X+0}|^2 \int_{t_{X+0}}^{t_{\rm had}} \mathrm{d}t_{X+1} \sum_i \int \frac{\mathrm{d}\Phi^{[i]}_{X+1}}{\mathrm{d}\Phi_X} \, \delta(t_{X+1} - t^{[i]}(\{p\}_{X+1})) A_i(...) \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+1})\right) \quad , \tag{20}$$

where the preceding (trivial) matching has been used to replace $w^{(R)}_{X+0}$ by $|M^{(0)}_{X+0}|^2$ and, as above, i runs over the discrete different branching possibilities, and A_i contains the explicit factor α_s. To match this to the tree-level $(X+1)$-parton matrix element, we first divide the complete matrix element phase space into a 'resolved' part at early (perturbative) times $t \leq t_{\rm had}$ corresponding to the region populated by the shower, and an unresolved part $t > t_{\rm had}$ which we will treat later. In the region with resolved perturbative radiation, we will compute the difference in the distribution of $\mathcal{O}$ by subtracting the shower term from that of the relevant tree-level matrix element (ME),

$$\text{ME} \quad : \quad \int_{t<t_{\rm had}} \mathrm{d}\Phi_{X+1} \, |M^{(0)}_{X+1}|^2 \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+1})\right) \quad . \tag{21}$$

Subtracting the parton shower term, the matching term (MT) at this partial order, differentially in the additional parton's phase space, is simply

$$\underbrace{|M^{(0)}_{X+1}|^2}_{\text{ME}} \; \underbrace{- \sum_{i \in X \to X+1} \Theta(t^{[i]}(\{p\}_{X+1}) - t_n) A_i(...) |M^{(0)}_{X+0}(\{\hat{p}_i\}_X)|^2}_{\text{PS}} \quad , \tag{22}$$

where i now runs over the number of possible contributing 'parent' configurations for the phase space point in question and the Θ function reflects the shower's not producing any jets harder than its starting scale (cf. e.g. the discussion of 'power' vs. 'wimpy' showers in ref. [36]). Above the starting scale, the shower effectively has a dead zone, and hence the matching term becomes just the unsubtracted matrix element. We show this mostly to illustrate the principle. Because the theta function only affects the hard emissions, and because the antenna radiation function captures all the infrared singularities of the matrix element, the subtracted matrix element is finite in all single-soft or simple collinear limits.

For final-state showers, we can start the shower at a nominally infinite resolution, $Q_H = 1/t_H \to \infty$, i.e. at $t = 0$, obviating the need for the explicit Θ function (we defer a discussion of initial-state showers to future work). We shall assume this to always be the case and thus define the equation for the matching term by:

$$\text{MT} \quad : \quad w^{(R)}_{X+1} = |M^{(0)}_{X+1}|^2 \; - \sum_{i \in X \to X+1} A_i(...) |M^{(0)}_{X+0}(\{\hat{p}_i\}_{X+0})|^2 \; . \tag{23}$$

As before, the hatted momenta $\hat{p}_i$ appearing in $M^{(0)}_{X+0}$ are a shorthand for the momenta obtained by operating on the $(X+1)$-parton configuration with an inverse map of type i,

$$\{\hat{p}_i\}_X = \{\kappa_i^{-1}(\{p\}_{X+1})\}_X \; . \tag{24}$$

From this discussion, it becomes clear how important it is for the phase-space map to allow a relatively clean phase space factorization such that the nested sums and integrals in eq. (20) produce a

manageable number of subtraction terms with simple borders. Below, we will construct the VINCIA showers explicitly with this goal in mind.

Before considering the unresolved and virtual corrections, let us remark on a few noteworthy aspects which appear already at this level. We noted above that leading-log resummation only fixes the soft/collinear singular terms of A. Variations in its finite terms are an implicit source of uncertainty and indeed can be used to help estimate this uncertainty (in addition to the standard techniques, such as varying the scale choice in α_s). We see here how the matching explicitly cancels such variations and hence reduces the uncertainty: if A is made "harder", then the shower generates more branchings, but the subtraction term in the matching equation also becomes larger, making the matching term smaller and compensating the change. Note also that, in the dipole-antenna formalism, there is no explicit dependence on the choice of evolution variable at this order, cf. eq.(23).

An extreme case arises if A is made so large that the matching term becomes negative in some region of phase space. This just means that the shower is overpopulating that region relative to the matrix element, and hence a negative correction is needed to counter-balance it. The corresponding correction events would have negative weights, but there is otherwise nothing abnormal about such a situation. Alternatively, one could switch to a shower re-weighting procedure as done in PYTHIA and thereby maintain positive event weights, but in the interest of simplicity we shall not consider reweighting in this paper.

The overall normalization of the parent process under study does change, by an amount given by the integral over the matching term. This includes an integral over the arbitrary finite terms in A. However, as we have yet to fix the corresponding virtual term, this just corresponds to changes within the range of tree-level uncertainty.

More importantly, this subtraction should in principle be easy to automate. Given any tree-level matrix element, which these days can be easily obtained from standard tools like ALPGEN [11], COMPHEP/CALCHEP [37], MADGRAPH [19], and others, the only additional ingredient needed is a subtraction term, whose most general form is a sum over lower-point matrix elements multiplied by evolution kernels. As mentioned above, because the leading singularities of the resulting subtracted matrix element are absent, it should be substantially easier to integrate efficiently over phase space than its unsubtracted counterparts.

Finally, we note that the matching scheme described above is inherently incremental. With presently available methods, a sample of unmatched events cannot easily be modified to produce a matched sample (except by doing sophisticated reweightings). Instead, a complete new sample must be generated using the matched generator. With our method, a pre-generated set of events need not be re-generated to improve the matching; we need only generate an additional set of events corresponding to the matching term in eq. (23) and add it to the first, with the relative weight of the two samples fixed by the relative integrated cross sections and the number of events in each sample. Of course, this only works to the extent that the particular A chosen for the first sample is known at the time the second set is generated. The total cross section corresponding to the combined sample would again be different than that of the original parton-shower sample, but the difference is of higher order.

3.3 Matching to $X + 0$ partons at loop level

To include the full $\mathcal{O}(\alpha_s)$ corrections to the initiator process X, and thereby fix the normalization of inclusive $(X + 0)$-parton observables (such as the inclusive cross section or the total width) to NLO, we

now turn to the relative order α_s corrections to X with zero additional (perturbatively resolved) partons. Again, the parton shower only generates one term, the first term in eq. (16) with the Sudakov (eqs. (2) & (11)) expanded to order α_s:

$$\text{PS} \quad : \quad -\int \mathrm{d}\Phi_X \, |M_X^{(0)}|^2 \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_X\right) \int_{t_X}^{t_{\mathrm{had}}} \mathrm{d}t_{X+1} \sum_i \int \frac{\mathrm{d}\Phi_{X+1}^{[i]}}{\mathrm{d}\Phi_X} \delta(t_{X+1} - t^{[i]}(\{p\}_{X+1})) A_i(...) \quad , \tag{25}$$

which, as a consequence of the unitary construction of the shower, is essentially identical to the real radiation term in eq. (20). The only differences are an overall minus sign, and the fact that the observable is here evaluated on the parent configuration (X) rather than on one with an additional emission ($X+1$).

We now wish to find the matching term that, together with the one for real radiation above, will give the full $\mathcal{O}(\alpha_s)$ corrections, possibly modulo power corrections in the non-perturbative cutoff t_{had}. To accomplish this, we need to include two terms from fixed-order matrix elements, one corresponding to genuine 1-loop corrections and another corresponding to the real radiation below the hadronization cutoff, which was left out above:

$$\text{ME} \quad : \quad \int_{t>t_{\mathrm{had}}} \mathrm{d}\Phi_{X+1} |M_{X+1}^{(0)}|^2 \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+1})\right) + \int \mathrm{d}\Phi_X 2\,\mathrm{Re}[M_X^{(0)} M_X^{(1)*}] \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_X)\right) \quad . \tag{26}$$

Let us re-emphasize that the extra parton in the first term is here unresolved (inclusively summed over) and hence the observable cannot really depend on it, up to an overall power correction. Within the required precision, the observable dependence is thus the same for all terms in eqs. (25) & (26), which we use to justify lumping them together below.

The matching term will again be defined by the remainder when subtracting off the parton shower contribution from the full matrix element. Differentially in $\mathrm{d}\Phi_X$ the matching term becomes:

$$\begin{aligned} \text{MT}: \quad w_X^{(V)} &= 2\,\mathrm{Re}[M_X^{(0)} M_X^{*(1)}] \; + \; |M_X^{(0)}|^2 \sum_i \int_{\mathrm{all}\,t} \frac{\mathrm{d}\Phi_{X+1}^{[i]}}{\mathrm{d}\Phi_X} A_i(...) + \int_{t>t_{\mathrm{had}}} \mathrm{d}\Phi_{X+1} w_{X+1}^{(R)} \\ &= 2\,\mathrm{Re}[M_X^{(0)} M_X^{*(1)}] \; + \; |M_X^{(0)}|^2 \sum_i \int_{\mathrm{all}\,t} \frac{\mathrm{d}\Phi_{X+1}^{[i]}}{\mathrm{d}\Phi_X} A_i(...) \; + \; \mathcal{O}(t_X/t_{\mathrm{had}}) \; , \end{aligned} \tag{27}$$

where we again used the properties of a clean phase space factorization and extended the definition of the subtracted w_{X+1} from eq. (23) into the unresolved region. Because the matched matrix element is free of soft or collinear singularities, the last term is just a power correction, below our required precision.

Note that a Θ function restricting the shower term to contribute only after t_X has again been avoided by letting the shower populate the entire phase space. In 1-loop matching with additional partons in the final state, a theta function similar to that in eq. (31) will be present. We defer a detailed discussion to future work.

As usual, the first two terms in eq. (27) are separately divergent and a regularization must be introduced before their (finite) sum can be evaluated. The divergences, which are universal, are usually regulated using dimensional regularization and the cancellation can be performed in a process-independent way. Only the finite terms must be computed anew for each new process. We thus believe that this part could also be automated fairly easily, once the required 1-loop matrix elements become available.

We see here how the NLO normalization of inclusive observables is fixed. In the matching of the real radiation term above, the LO normalization changed by the integral of $w^{(R)}$, a quantity which depends

explicitly on the finite terms in A_i. The same variation is subtracted in eq. (25). The final normalization should accordingly be stable up to higher-order and non-perturbative power corrections.

3.4 Matching to $X+2$ partons (and beyond) at tree level

So far, we have discussed a matching prescription similar to that of the program MC@NLO, though we have here attempted to develop a formalism more readily applicable to the treatment of non-collinear ambiguities and associated uncertainties. The next step in reducing these is to include further information from tree-level coefficients deeper in the perturbative series.

As mentioned in section 2, we shall now limit our attention to evolution variables which fulfill the Markov property in the strictest sense, i.e. which do not have any explicit memory of the event history. It then becomes irrelevant whether a particular $(X+1)$-parton configuration was obtained by parton showering from $X+0$ partons or from the tree-level $(X+1)$-parton matching term. With a uniquely defined "restart scale" t_{X+1} in both cases, the subsequent evolution also becomes the same.

In fact, the Markov property solves nearly the entire problem for us. We are interested in the relative order α_s^2 double real radiation term ($X+2$ partons) from a shower which we assume has already been matched to $X+1$ partons above. By virtue of this prior matching, the total $(X+2)$-parton contribution, for a history-independent evolution variable, is just given by the parton shower off the tree-level $(X+1)$-parton matrix element, here differentially in $(X+1)$-parton phase space:

$$\text{PS} \quad : \quad |M^{(0)}_{X+1}|^2 \int_{t_{X+1}}^{t_{\rm had}} \mathrm{d}t_{X+2} \sum_i \frac{\mathrm{d}\Phi^{[i]}_{X+2}}{\mathrm{d}\Phi_{X+1}}\, \delta(t_{X+2} - t(\{p\}_{X+2}))A_i(...)\delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+2})\right) \quad , \tag{28}$$

which, apart from the replacement $X \to X+1$ and the restriction that t not depend on i (the Markov property), is identical to the expression in eq. (20). The second-level matching term would then be,

$$|w^{(R)}_{X+2}|^2 = |M^{(0)}_{X+2}|^2 - \sum_i A_i(...)|M^{(0)}_{X+1}|^2 \Theta(t(\{p\}_{X+2}) - t(\{\hat{p}\})_{X+1}) \quad , \tag{29}$$

where the momentum configuration $\{\hat{p}\}_{X+1}$ is defined by eq. (24) and the Θ function expresses that the shower evolution is ordered, i.e. that t_{X+2} must come after t_{X+1}. That is, the matching coefficient w is obtained precisely by subtracting the leading singularities as expressed by the evolution kernel (including also the finite terms). It is essentially the same as the subtraction term that would be used for real emission in a next-to-leading order calculation, apart from the Θ function which here imposes that nested antennae be kinematically ordered, as measured by the shower evolution variable.

Because of the Markov property, this procedure can be repeated for tree-level matrix elements with an arbitrary number of additional emissions. However, while the LL antenna functions capture all the leading singularities, the full higher-order matrix elements will generally contain sub-leading singularities as well. This leads to problems with unwanted contributions coming from matrix elements with "too many" final-state partons. In general, all the following terms may appear (after integrating over phase space),

$$\alpha_s^n \prod_{m=0}^{2n} L_{nm} \ln^{2n-m}(Q_1^2/Q_2^2) \; , \tag{30}$$

where $Q_{1,2}$ are scales in the problem and L_{mn} are finite coefficients. For example, at each order n, $m=0$ is the double logarithmic (eikonal) term and $m=2n$ is the non-logarithmic (" finite") one.

We can now be more specific. Since the shower generates only the (leading-color) $m = 0$ and $m = 1$ pieces exactly, the subtraction in eq. (29), beyond $n = 1$, may leave pieces inside the matching term which would be divergent were it not for the hadronization cutoff. If left alone, the excess contributions with $m = 2$ and greater would lead to distributions of physical quantities with overly large subleading log contributions (divergent in the limit the hadronization cutoff is removed), which is obviously not desirable.

At $n = 2$, corresponding to $(X + 2)$-parton matching at tree level, these divergences would be removed in an NLL shower (where "next-to-leading" here means with respect to the LL shower). Though we do make some remarks aimed in this direction at the end of the paper, we note that even if we were able to present a complete solution, the same problem would then just appear at NNLL level when attempting $(X + 3)$-parton matching, and so on. To do tree-level matching beyond one additional parton, clearly, we need a prescription to consistently regulate the subleading logarithms in tree-level $(X + n)$-parton matrix elements. The uncertainty they induce is within the stated accuracy of the calculation, as they are higher order in both logarithms and powers of the strong coupling.

One possibility is to nominally subtract the subleading logarithms as well in eq. (29), to the extent they are known. Although the LL shower wouldn't regenerate them, this procedure would at least cure the problem without affecting the validity of the approach, up to subleading logarithmic corrections. However, for tree-level matching to X + many partons, the analytic form of all the corresponding $\mathrm{N}^{\mathrm{many}}$LL terms would then have to be explicitly subtracted, clearly overkill considering that all we are really after is just a regulator.

A simpler approach is to place explicit restrictions on the w_{X+2} phase space, cutting out the regions where the subleading logarithms become important, for instance by introducing cuts on parton–parton invariant masses or transverse momenta. (This would be equivalent to the "matching scale" cut imposed in CKKW approaches.) As a rule of thumb one should choose the cut to be smaller than the hard scale Q_X (so as not to disturb the matching in the hard region) but still sufficiently large that $\ln(Q_X^2/Q_{\mathrm{cut}}^2)$ is not much greater than unity. A back-of-the-envelope estimate would be that roughly one order of magnitude between the two scales could be a reasonable starting point.

Finally, an alternative approach is to use a Sudakov or Sudakov-like function as a regulating factor. This smoothly suppresses unwanted configurations while simultaneously maintaining a fixed-order expansion that begins at unity over all of phase space.

This gives the following general form for leading-order matching with any number of additional partons

$$\begin{aligned} \mathrm{MT} \quad : \quad w^{(R)}_{X+1+n} \quad &= \quad \tilde{\Delta}(\{p\}_{X+1+n})\Big(|M^{(0)}_{X+1+n}|^2 \\ &\quad - \sum_i A_i(...)|M^{(0)}_{X+n}|^2\Theta(t(\{p\}_{X+n+1}) - t(\{\hat{p}\})_{X+n})\Big) \ ; \ t < t_{\mathrm{had}} \ , \end{aligned} \tag{31}$$

where $\tilde{\Delta}$ is either the Sudakov-like function just mentioned or, alternatively, just a Θ function for the cut-off case mentioned above.

For automated approaches, the Θ function method is probably more appropriate for a stand-alone matrix element generator, which would not have the shower Sudakov readily available, whereas a more integrated solution could more easily make use of the smoother Sudakov suppression.

How does this work in practice? To generate a sample of events matched to n additional partons at tree level, we should generate events with zero through n partons according to probabilities given by

the subtracted matrix elements of eq. (31), and then evolve each configuration using the parton shower, down to $Q_{\rm had}$.

4 Shower Termination

The traditional approach in Monte Carlo parton-shower generators is to cut the shower evolution off at a low value of the evolution scale, $Q_{\rm had}$ of $\mathcal{O}$(1 GeV). At this point a transition to a different "evolution" is made, in the form of QCD-inspired phenomenological hadronization models which explicitly enforce confinement and other non-perturbative features.

From the point of view of a perturbative calculation, this cutoff is simply an arbitrary infrared regulator, below which partons are not resolved. In the context of the ordered evolution of parton showers, however, it represents a scale at which non-perturbative components of the evolution become significant, and hence at which point the perturbative evolution kernels used in the parton-shower approximation no longer suffice to describe the physics of events; that is, the "evolution" should really contain large corrections e.g. from pion resonances.

In the context of the VINCIA code, we start by defining the infrared cutoff $Q_{\rm had}$ in a more universal way. Because it simply represents a separation between regions with "large" and "small" non-perturbative corrections, respectively, it is *not* necessary to tie it to the perturbative evolution variable. Any infrared-safe phase space contour will do. For instance, one could easily imagine defining a hadronization cutoff in terms of dipole-antenna masses applied to a shower which uses transverse momentum as its evolution variable, as long as the former regulates all perturbative divergences and the latter separates off all regions where hadronization corrections are expected to be large.

We denote the phase space contour defining the hadronization cutoff for an n-parton configuration by $\Theta_{\rm had}(\{p\}_n)$:

$$\Theta_{\rm had}(\{p\}_n) = \Theta(t_{\rm had} - t_{\rm had}(\{p\}_n)) = \begin{cases} 1 & \text{in "perturbative" region} \\ 0 & \text{in "non-perturbative" region} \end{cases} \tag{32}$$

where $t_{\rm had} = 1/Q_{\rm had}$ is the value of the hadronization cut-off and its functional form (which may be different from that of the evolution variable) is given by $t_{\rm had}(\{p\})$. The Sudakov factor then takes the form,

$$\Delta(t_n, t_{\rm end}; t_{\rm had}) = \prod_{i\in\{n\to n+1\}} \exp\left(-\int_{t_n}^{t_{\rm end}} {\rm d}t_{n+1} \int \frac{{\rm d}\Phi_{n+1}^{[i]}}{{\rm d}\Phi_n} \delta(t_{n+1} - t(\{p\}_{n+1}))\Theta_{\rm had}(\{p\}_{n+1}) A_i(...)\right) \; ; \tag{33}$$

for brevity, we have rewritten the sum over i in eq. (2) in product form. The perturbative shower termination scale $t_{\rm end}$ can now be taken to infinity without any problem, as the divergences are explicitly regulated by $\Theta_{\rm had}$. The probability that the configuration emits no perturbative (resolved) radiation at all is,

$$\Delta(t_n, \infty; t_{\rm had}) = \prod_{i\in\{n\to n+1\}} \exp\left(-\int_{t_n}^{\infty} {\rm d}t_{n+1} \int \frac{{\rm d}\Phi_{n+1}^{[i]}}{{\rm d}\Phi_n} \delta(t_{n+1} - t(\{p\}_{n+1}))\Theta_{\rm had}(\{p\}_{n+1}) A_i(...)\right) \; , \tag{34}$$

corresponding to $Q_{\rm end} = 1/t_{\rm end} \to 0$. This probability is non-vanishing. The matching equations in section 3 remain unaltered by the introduction of this hadronization cut-off, except for the replacements $t_{\rm end} \to \infty$ in the integral boundaries along with $A \to \Theta_{\rm had} A$.

The hadronization cut-off has traditionally been imposed in terms of the evolution variable itself, since, getting one job done well, it usually gets the other done almost as well. (A few additional cutoffs are normally imposed, e.g. to avoid systems with very low invariant masses, but those are minor points). This has the disadvantage of making the region defined to be "non-perturbative" different from shower model to shower model, and hence a hadronization model fitted with one shower cannot be used as is with any other shower.

Decoupling the form of the hadronization cut-off, as proposed here (34), from the shower parameters (and in particular the evolution variable), would make the non-perturbative modeling more universally applicable. This should be true up to the uncertainty inherent in the perturbative evolution itself.

This would also be a step towards making it meaningful to compare different parton showers before hadronization. This is in stark contrast to the present situation, where different parton showers are far from directly comparable, each having its own cut-off along its own contour. Fixed-order parton-level calculations could then be replaced by parton showers not including hadronization and matched to fixed order matrix elements as the "gold standard" of what is a good perturbative QCD calculation.

5 The VINCIA Code

We now turn to a proof-of-concept implementation of the ideas contained in previous sections, in the form of the VINCIA code (Virtual Numerical Collider with Interleaved Antennae), implemented both as a stand-alone program and as a final-state shower plug-in for PYTHIA8. We have also implemented matching of $H \to gg$ to first order at both tree and 1-loop level, according to the matching terms defined in section 3. For the plug-in, this includes the possibility of generating negative-weight correction events when the shower is overpopulating phase space.

The numerically implemented shower is based on an interleaved evolution (see e.g. ref. [3]) of systems of color-ordered QCD antennae. The implementation discussed here is limited to gluons, and uses a strict dipole-antenna factorization [30]. (The name 'dipole factorization' is associated with a related NLO formalism due to Catani and Seymour [38].) Inserting the massless dipole phase space, eq. (10), into the event Sudakov eq. (34) yields a product of individual color-ordered dipole Sudakov factors

$$\Delta(t_{\rm n}, t_{\rm end}; \{p\}_n, t_{\rm had}) = \prod_i^n \exp\left(-\int_{t_n}^{\infty} {\rm d}t_{n+1} \int_0^s {\rm d}s_{ar} \int_0^{1-s_{ar}} {\rm d}s_{rb} \int_0^{2\pi} \frac{{\rm d}\phi}{2\pi} \delta(t_{n+1} - t(\{p\}_{n+1})) \Theta_{\rm had}(\{p\}_{n+1}) \frac{A_i(...)}{16\pi^2 s^{[i]}}\right) , \tag{35}$$

where the branching invariants $s_{ar} = (p_a + p_r)^2$ and $s_{rb} = (p_r + p_b)^2$ are illustrated in Fig. 1. We now proceed to give explicit forms for each of the objects required by section 2 for the definition of a shower.

5.1 Evolution Variable

We shall here consider only Lorentz-invariant evolution variables, $Q_E = 1/t$. We have implemented two different choices, corresponding to ordering in transverse momentum and in dipole mass (related,

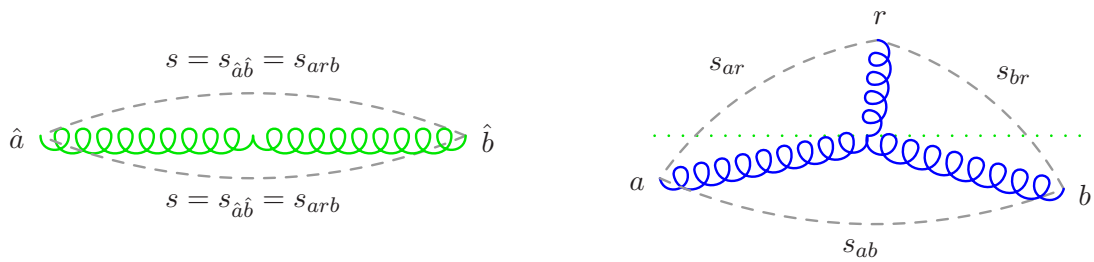

Figure 1: Left: illustration of the two original dipole antennae in a closed color-singlet gg system in the center-of-mass. Right: the system after one branching, showing the branching phase space invariants s_{ar} and s_{rb}. The ϕ angle corresponds to rotations around the axis of the original dipole. The ψ angle corresponds to a rotation of the branching system about an axis perpendicular to the branching plane. The "forbidden angle" θ, always set to zero in our maps below, would correspond to rotating the branching plane off axis with respect to the original dipole.

but not identical, to parton virtuality) respectively:

$$Q_E^2 = \begin{cases} \text{type I} \;:\; Q_{\rm I}^2 \equiv 4\frac{s_{ar}s_{rb}}{s} = 4p_\perp^2{}_{\textsc{Ariadne}} \\ \text{type II} \;:\; Q_{\rm II}^2 \equiv 2\min(s_{ar}, s_{rb}) \end{cases} , \tag{36}$$

where the normalizations have been chosen so that the maximum value of the evolution variable is always the dipole-antenna invariant mass s (to avoid cluttering the notation, we now let the superscript $[i]$ be implicit). We will usually work with dimensionless versions of these invariants,

$$y_E = \begin{cases} \text{type I} \;:\; y_{\rm I}^2 = \frac{Q_{\rm I}^2}{s} = 4\frac{s_{ar}s_{rb}}{s^2} = 4y_{ar}y_{rb} \\ \text{type II} \;:\; y_{\rm II}^2 = \frac{Q_{\rm II}^2}{s} = 2\min(y_{ar}, y_{rb}) \end{cases} , \tag{37}$$

where $y_i = s_i/s$, so that the maximal value of $y_{\rm I,II}$ inside the physical phase space is unity. A comparison of iso-y contours for these two variables in the branching phase space is shown in Fig. 2. Their complementary nature is now readily apparent. The transverse-momentum or ARIADNE variable (type I) will categorize a hard but collinear branching (close to one of the axes) as harder than a wide-angle but soft one (close to the origin), whereas the the dipole-mass ordering (type II) will tend to do the opposite. This affects which regions act to Sudakov suppress which branchings during the evolution. Because the antenna function A captures the singularities of the matrix element completely, and because using either of these evolution variables integrates the leading singularities, thereby capturing the correct leading-log behavior, they are both appropriate choices for evolution variables. (Further choices beyond the two above are possible as well.) Likewise, the coherent dipole radiation pattern approximated by angular ordering in conventional parton showers are captured in the antenna function (see, e.g., [15]), and hence coherence is independent of the choice of evolution variable to first order in this formulation. We therefore do not include an "angular ordering" option.

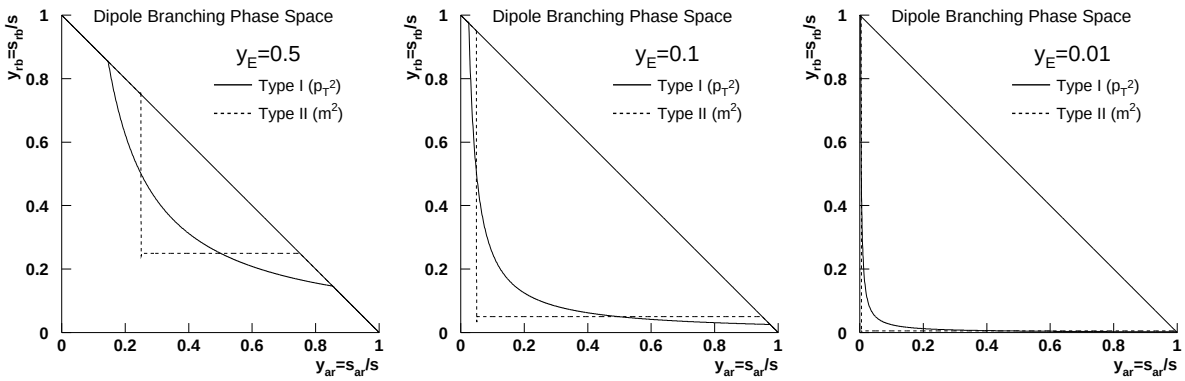

Figure 2: Illustration of the progression of the evolution variables over phase space for massless partons: contours corresponding to $y_{\mathrm{I,II}}$ =0.5(left), 0.1(middle), and 0.01(right). On the x and y axes respectively are the rescaled invariants $y_{ar} = s_{ar}/s$ and $y_{rb} = s_{rb}/s$ after the branching. The soft singularity is at the lower left corner, and the collinear singularities lie along the axes.

Note also that the definitions in eq. (37) do not yet completely obey the Markov condition. Because gluons are indistinguishable, it is not possible to single out the radiated parton r without knowing the branching history of the configuration. In other words, when showering off a three-gluon configuration with an unspecified history (e.g. from the three-gluon matching term), we have several possible choices of what "restart" scale to choose, depending on which of the partons we decide to call r. We emphasize that this is not a problem for matching to first order ($X+1$ at tree level and $X+0$ at loop level), since the history-dependence only has to do with what *restart* scale to choose and hence, at the earliest, affects the second emission. There is no fundamental difficulty in defining variables which strictly obey the Markov condition, but as already discussed in section 3, we postpone a detailed discussion of this aspect to future work.

5.2 Phase Space Map

We must next choose a phase space map. The restriction to dipole-antenna phase space factorization, eqs. (5) & (10), already fixes most of the δ functions: all partons except the two involved in the splitting are just "copied" to the $(n+1)$ configuration. Denoting the branching antenna pair by $[i]$, eq. (6) implies

$$p_j = \hat{p}_j \quad \forall \quad j \notin [i] \; . \tag{38}$$

The branching antenna pair, denoted $\hat{a}$ and $\hat{b}$, are replaced as shown in fig. 1, by a trio of partons, denoted a, b, and r. This replacement conserves energy and momentum, and keeps all partons at their physical masses. In the center of mass frame of the parent dipole, the energies are related to the branching invariants as follows,

$$\begin{aligned} E_a &= \tfrac{s-s_{rb}+m_a^2}{2\sqrt{s}} \; , \\ E_b &= \tfrac{s-s_{ar}+m_b^2}{2\sqrt{s}} \; , \\ E_r &= \tfrac{s-s_{ab}+m_r^2}{2\sqrt{s}} \; . \end{aligned} \tag{39}$$

Our discussion here will focus on massless partons, $m_a = m_b = m_r = 0$. Since the phase space construction implicitly uses δ functions requiring the partons to be on shell, the absolute values of the momenta are equal to the energies for massless particles.

Staying in the dipole's center-of-mass frame (DCM), there is no freedom left to choose the relative angles between the three daughter partons in the branching plane:

$$\begin{aligned} \cos\theta_{ar} &= \frac{2E_aE_r + m_a^2 + m_r^2 - s_{ar}}{2|\mathbf{p}_a||\mathbf{p}_r|} \ , \\ \cos\theta_{ab} &= \frac{2E_aE_b + m_a^2 + m_b^2 - s_{ab}}{2|\mathbf{p}_a||\mathbf{p}_b|} \ . \end{aligned} \tag{40}$$

We still need to fix the orientation of the three daughter partons with respect to the parent dipole. This involves three Euler angles. As noted in section 2, one of these is fixed by requiring that the branching plane contains the dipole axis, or equivalently that the normal to the plane be orthogonal to the dipole axis. (This imposes parity conservation on the $2 \to 3$ transition.) Another angle is just the integration variable ϕ representing rotations around the dipole axis. While the latter is here chosen isotropically we note that the matching terms will still (re-)introduce anisotropies up to the order of the matching.

The remaining ambiguity in the phase space map thus rests entirely with the last Euler angle, the one corresponding to rotations around an axis perpendicular to the branching plane. In the context of ARIADNE [17], a choice was made which "least disturbed" neighboring dipoles[4]. In order to explore the consequences of this ambiguity, we have so far implemented three discrete possibilities for this angle, defined by eq. (8) to be the angle between parton a and the original parton $\hat{a}$ in the DCM frame,

$$\psi_{\text{ARIADNE}} = \frac{E_b^2}{E_a^2 + E_b^2}(\pi - \theta_{ab}) \tag{41}$$

$$\psi_{\text{PS}} = \left\{ \begin{array}{ll} 0 & ; s_{ar} > s_{rb} \\ \pi - \theta_{ab} & ; s_{ar} < s_{rb} \end{array} \right\} \tag{42}$$

$$\psi_{\text{Ant}} = 1 + \frac{2y_{aa}}{1 - y_{rb}} \ , \tag{43}$$

where y_{aa} in the last line is defined by:

$$f = \frac{y_{rb}}{y_{ar} + y_{rb}} \ , \tag{44}$$

$$\rho = \sqrt{1 + 4f(1-f)y_{ar}y_{rb}/y_{ab}} \ , \tag{45}$$

$$y_{aa} = -\frac{(1-\rho)y_{ab} + 2fy_{ar}y_{rb}}{2(1 - y_{ar})} \ . \tag{46}$$

It is important on physical grounds that $\psi \to 0$ smoothly as parton r becomes collinear with parton b (that is, as $s_{rb} \to 0$), so that parton a becomes aligned with parent parton $\hat{a}$, and likewise when the roles of a and b and $\hat{a}$ and $\hat{b}$ are simultaneously interchanged. This ensures that the daughter system approaches the parent one in this limit. Otherwise, however, there are no constraints on ψ. All three alternatives satisfy this constraint.

[4]For quark antennae, Kleiss has shown that an optimal choice exists [35], but for gluon antennae the situation is less clear.

The first choice corresponds to the ARIADNE map, just as our first evolution variable corresponds to the ARIADNE one [17]. We hope this helps make comparisons between the two approaches simpler. The second corresponds roughly to conventional parton showers, in which the non-radiating parton only recoils longitudinally. (Since our antenna shower does not maintain a clear distinction between which parton radiates, the one with the largest invariant mass with respect to r is chosen to play the part of recoiler.) The last choice is an example of a more general form; different choices of f could be used to explore it more fully.

5.3 Radiation Function

We have thus far implemented only the $gg \to ggg$ radiation function, for which we have used the Gehrmann–Gehrmann-De Ridder–Glover "global" antenna function f_3^0 [39][5],

$$f_3^0(p_a, p_r, p_b) = \frac{1}{s^{[i]}}\left[(1 - y_{ar} - y_{rb})\Bigg(\underbrace{\frac{2}{y_{ar}y_{rb}}}_{\text{“soft”}} + \underbrace{\frac{y_{ar}}{y_{rb}} + \frac{y_{rb}}{y_{ar}}}_{\text{“collinear”}}\Bigg) + \frac{8}{3}\right]. \tag{47}$$

In this formula, $s^{[i]}$ is the mass squared of the dipole-antenna. "Global" means that the phase space of each antenna is unrestricted by overlap with other antennae; the normalization and singularities are such that the sum of contributions has the desired structure. The leading (double logarithmic) singularities correspond to two invariants vanishing (soft radiation), and arise only from one antenna. The single-logarithmic (collinear) singularities receive contributions from two neighboring antenna.

We choose a second-order polynomial in the invariants for the form of the arbitrary finite terms, imposing only the restriction that the antenna function be positive definite. Combining f_3^0 above with the normalization implied by eq. (35), the radiation function for the VINCIA gluon shower becomes:

$$A(p_a, p_r, p_b) = \frac{4\pi\alpha_s(\mu_R)\, N_c}{s^{[i]}}\left[(1 - y_{ar} - y_{rb})\left(\frac{2}{y_{ar}y_{rb}} + \frac{y_{ar}}{y_{rb}} + \frac{y_{rb}}{y_{ar}}\right) + \sum_{\alpha,\beta\geq 0} C_{\alpha\beta}\, y_{ar}^{\alpha} y_{rb}^{\beta}\right], \tag{48}$$

where finite terms are parametrized by the constants $C_{\alpha\beta}$. We can explore systematically the consequences of making the radiation function harder or softer by varying $C_{\alpha\beta}$; e.g., the special case corresponding to the f_3^0 antenna function can be obtained by choosing $C_{00} = 8/3$. As discussed in section 3, matching absorbs these variations in the matching terms, leaving only the uncertainty due to genuine higher-order terms in the shower. We can thereby quantify the reduction in the associated uncertainty.

5.4 Renormalization Scale

We let the renormalization scale for α_s at each branching be given either by the evolution scale at the branching, Q_E^2, or by the invariant mass of the dipole being evolved, $s^{[i]}$,

$$\mu_R = \begin{cases} \text{type 1} \;:\; \mu_1 = K_R Q_E \\ \text{type 2} \;:\; \mu_2 = K_R\sqrt{s^{[i]}} \end{cases}, \tag{49}$$

[5]Note that we have changed the non-singular term from 2/3 to 8/3, relative to the original paper.

where Q_E is the evolution variable and we allow for an arbitrary prefactor K_R to be applied. For example a factor $K_R = 1/2$ applied to the type I evolution variable would yield a renormalization scale equal to the ARIADNE definition of transverse momentum. By default we will use a 1-loop running α_s, but we leave open the option of studying fixed coupling or two-loop running as well[6].

5.5 Starting and Ending Scales

For a parent process producing two partons in a decay, such as $H \to gg$ which we shall consider below, we choose the initial starting scale to be the full phase space, s, so that the shower does not have a dead zone. After branching at scale Q_E, the shower evolution continues from that scale. As already discussed, this does imply a slight dependence on the shower history, as the same configuration can in principle be obtained by different branchings corresponding to different values of Q_E. For showers off the three-gluon matching term, which has no history and thus no unique Q_E attached to it, we compute the scale corresponding to each possible ordering and select the smallest of these, as the matching is intended to describe the hardest emission. The shower is cut off in the infrared by an evolution-independent contour, as described in section 4. The choices possible for the functional form of this contour are the same as for the evolution variable, eq. (36). The history dependence could be eliminated by implementing a true Markov evolution; we leave further discussion of this to future work.

5.6 Shower Implementation

Shower generation proceeds as follows. Given a starting scale Q_n, a trial branching for each antenna dipole is found by generating a random number $R \in [0, 1]$ and solving for Q_{n+1} in the following "trial equation":

$$\begin{aligned} R &= \widehat{\Delta}(Q_n, Q_{n+1}) \\ &= \exp\left[-\int_{Q_{n+1}}^{Q_n} \mathrm{d}Q_{n+1} \int_0^s \mathrm{d}s_{ar} \int_0^{s-s_{ar}} \mathrm{d}s_{rb}\, \delta(Q_{n+1} - Q(\{p\}_{n+1})) \frac{\hat{A}(...)}{16\pi^2 s^{[i]}} \right], \end{aligned} \tag{50}$$

where we use $\widehat{\Delta}$ to signify that a nominally larger branching probability $\hat{A} > A$ may be used to generate these trials (for instance using an over-estimate of $\alpha_s^{\max} > \alpha_s(\mu_R)$, no hadronization cutoff, etc); the resulting distribution will then be corrected by subsequent vetos.

In traditional approaches, an equation for $Q_{n+1}(Q_n, R)$ is obtained by analytically inverting eq. (50). Since we wish to be able to choose arbitrary evolution variables and radiation functions, however, we have instead used a more numerical approach.

For fixed coupling, the Sudakov factor only depends on one quantity, the ratio of the evolution scale Q_{n+1} to the starting scale Q_n. Re-expressing the Sudakov factor in terms of dimensionless ratios of invariants,

$$\begin{aligned} \widehat{\Delta}(y_{\mathrm{trial}}) &\equiv \widehat{\Delta}(1, y_{\mathrm{trial}}) \\ &= \exp\left[-\int_{y_{\mathrm{trial}}}^1 \mathrm{d}y_E \int_0^1 \mathrm{d}y_{ar} \int_0^{1-y_{ar}} \mathrm{d}y_{rb}\, \delta(y_E - y_E(y_{ar}, y_{rb})) \frac{s^{[i]}\hat{A}(...)}{16\pi^2} \right], \end{aligned} \tag{51}$$

[6]In the PYTHIA8 plug-in, we rely on the α_s implementation in PYTHIA, which likewise provides these choices.

where $y_E = Q_E$ is the dimensionless evolution scale, as defined by eq. (37). Because the combination $s^{[i]}\hat{A}(...)$ is independent of $s^{[i]}$, cf. eq. (48), this quantity depends only on a single variable, $y_{\rm trial}$. Accordingly, it is simple to tabulate it during initialization; we do so using a cubic spline, performing the integrals inside the exponent either numerically, via two-dimensional adaptive Gaussian quadrature, or analytically, as a counter-check. (Using a running coupling constant introduces a dependence on explicit scales, which we implement using a veto as described below.)

We may then solve the equation $R = \widehat{\Delta}(y_{\rm trial})$ numerically for $y_{\rm trial}$ using the splined version of the Sudakov and standard root finding techniques. These are computationally quite efficient.

The antenna with the largest trial scale is then selected for further inspection. A ϕ angle is chosen uniformly, and the remaining degeneracy along the iso-y contour (as shown for example in fig. 2) is lifted by choosing a complementary invariant, which we call z, according to the probability distribution,

$$\begin{aligned} I_z(y_E, z) &= \int_0^z \mathrm{d}z' \int_0^1 \mathrm{d}y_{ar} \int_0^{1-y_{ar}} \mathrm{d}y_{rb}\, \delta(y_E - y_E(y_{ar}, y_{rb}))\, \delta(z' - z(y_{ar}, y_{rb})) \frac{s^{[i]}\hat{A}(...)}{16\pi^2} \\ &= \int_{z_{\min}(y_E)}^z \mathrm{d}z' |J(y_E, z')| \frac{s^{[i]}\hat{A}(...)}{16\pi^2} \quad , \end{aligned} \tag{52}$$

where $|J(y_E, z)|$ is the Jacobian arising from translating $\{y_{ar}, y_{rb}\}$ to $\{y_E, z\}$ and $z_{\min}(y_E)$ is the smallest value z attains inside the physical phase space for a given y_E. Since z merely serves as a parametrization of phase space along an iso-y_E contour, its definition is arbitrary, so long as it is linearly independent of y_E. Depending on the type of evolution variable, we choose z as

$$\begin{aligned} &\text{type I} \quad : \quad z = y_{rb} \quad , \\ &\text{type II} \quad : \quad z = \max(y_{ar}, y_{rb}) \quad , \end{aligned} \tag{53}$$

leading to the Jacobian factors $|J_{\rm I}| = 1/(4z)$ and $|J_{\rm II}| = 1/2$, respectively, and the phase space boundaries

$$\begin{aligned} &\text{type I} \quad : \quad z_{\min}(y_E) = \tfrac{1}{2}(1 - \sqrt{1 - y_E}) \quad , \quad z_{\max}(y_E) = \tfrac{1}{2}(1 + \sqrt{1 - y_E}) \quad , \\ &\text{type II} \quad : \quad z_{\min}(y_E) = \tfrac{1}{2} y_E \quad , \quad z_{\max}(y_E) = 1 - \tfrac{1}{2} y_E \quad , \end{aligned} \tag{54}$$

where the type II case should be divided into two branches, one with $y_{ar} > y_{rb}$ and one with $y_{rb} > y_{ar}$, each having the phase space limits given here.

Because the I_z functions depend on two independent variables, y_E and z, we have not implemented a splined approach for this task. Instead, we use analytical integrals over the two kinds of phase space regions we are interested in. In generic form, these are

$$I_z(y_E, z) = \frac{\alpha_s^{\max} N_c}{4\pi} \left[S(z) - S(z_{\min}) + K(z) - K(z_{\min}) \right] \ , \tag{55}$$

with $S(z)$ coming from the soft and collinear singular terms,

$$\begin{aligned} &\text{type I} \quad : \quad S_{\rm I}(z) = \frac{y_E^2}{192 z^3} - \frac{y_E}{32 z^2} + \frac{8 + y_E}{16 z} - \frac{z}{4} + \frac{-12z + 3z^2 - 2z^3 + 12 \ln z}{6 y_E} \ , \\ &\text{type II} \quad : \quad S_{\rm II}(z) = \frac{1}{24 y_E} \left(-6(8 + y_E^2 + 2z - z y_E) z - 8z^3 + 3(2 - y_E)(8 + y_E^2) \ln z \right) \ , \end{aligned} \tag{56}$$

and $K(z)$ coming from the finite polynomial,

$$\begin{aligned}
\text{type I} \quad : \quad K_{\rm I}(z) &= \frac{1}{4}\sum_{\alpha}\left(\frac{y_E}{4}\right)^{\alpha}\left[C_{\alpha\alpha}\ln z + \sum_{\beta\neq\alpha} C_{\alpha\beta}\frac{z^{\beta-\alpha}}{\beta-\alpha}\right] , \\
\text{type II} \quad : \quad K_{\rm II}(z) &= \frac{1}{2}\sum_{\alpha,\beta} C_{\alpha\beta}\frac{z^{\beta+1}}{\beta+1}\left(\frac{y_E}{2}\right)^{\alpha} \quad (\, C_{\alpha\beta} \leftrightarrow C_{\beta\alpha} \text{ for } y_{ar} > y_{rb}) \, ,
\end{aligned} \tag{57}$$

with $\alpha, \beta \geq 0$.

With the trial resolution scale y_E and the energy-sharing fraction z now in hand, we can compute y_{ar} and y_{rb}. Together with ϕ (chosen above), this gives the complete branching kinematics. We now apply a veto, accepting the trial branching with probability,

$$P_{\rm accept} = \Theta_{\rm had}(Q_{\rm had}(\{p\}_{n+1}) - Q_{\rm had})\frac{\alpha_s(\mu_R(\{p\}_{n+1}))}{\alpha_s^{\rm max}} \, . \tag{58}$$

That is, the branching is only accepted if it is inside the perturbative region and then only with probability $\alpha_s/\alpha_s^{\rm max}$, which reduces the effective coupling to the correct value by virtue of the veto algorithm. (Note that the event is not thrown away, it is merely the branching which is vetoed.) In order to evolve the system further, we repeat the steps above. The trial branching scale becomes the new starting scale, whether the above branching was vetoed or not.

The evolution continues until there is no perturbative evolution space left (the equivalent of reaching the hadronization cutoff in our terminology). In the current implementation, we consider this condition satisfied for a given antenna if ten consecutive trials are rejected due to the $\Theta_{\rm had}$ condition.

5.7 Matching Implementation

In this paper, we restrict ourselves to matching at first order (at tree and loop level) for a scalar decaying into two gluons via an effective point coupling. By first order matching we mean that, in addition to $w_2^R = |M_2^{(0)}|^2$, we include the matching coefficients $w_3^{(R)}$ and $w_2^{(V)}$. For the decay process $H \to gg$ the subtracted matrix elements are relatively easy to obtain. Given the Born squared matrix element $|M_2^{(0)}|^2$ for $H \to gg$ we find for the 2-gluon matching term at one loop

$$w_2^{(V)} = \frac{\alpha_s(\mu_R)N_c}{2\pi}\left[\frac{87}{6} + \frac{11}{3}\log\left(\frac{\mu_R^2}{M_H^2}\right) + \sum_{\alpha,\beta} C_{\alpha\beta}\frac{\alpha!\beta!}{(2+\alpha+\beta)!}\right]|M_2^{(0)}|^2 \tag{59}$$

and for the 3-gluon matching term

$$\begin{aligned}
w_3^{(R)} &= \frac{8\pi\alpha_s(\mu_R)N_c}{M_H^2}\Big(8 - F_{123} - F_{231} - F_{312}\Big)|M_2^{(0)}|^2 \\
&= \frac{8\pi\alpha_s(\mu_R)N_c}{M_H^2}\Big(8 - 3C_{00} - (C_{10}+C_{01}) - C_{11}\left(y_{12}y_{23} + y_{23}y_{31} + y_{31}y_{12}\right) + \cdots\Big)|M_2^{(0)}|^2 \, ,
\end{aligned} \tag{60}$$

where

$$F_{arb} = \sum_{\alpha,\beta\geq 0} C_{\alpha\beta}y_{ar}^{\alpha}y_{rb}^{\beta} \, . \tag{61}$$

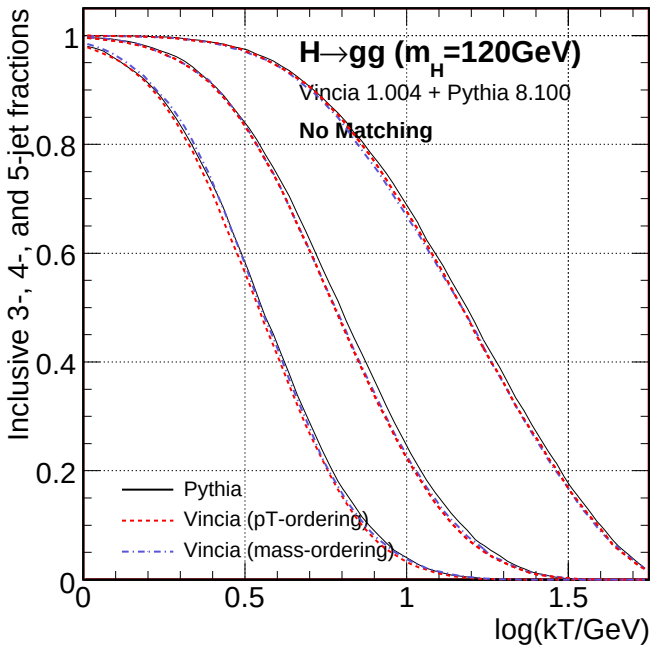

Figure 3: Comparison between the PYTHIA8 and VINCIA showers, with as similar settings as possible. Inclusive 3-, 4-, and 5-jet rates for fully showered parton-level events as a function of Durham k_T. For VINCIA, two different choices of the evolution variable are shown. For PYTHIA, $g \to q\bar{q}$ was switched off, as was gluon polarization effects. All curves use a 1-loop running coupling with $\alpha_s(m_Z) = 0.12$. The $p_\perp$ definition of the respective generator is used as both renormalization scale in α_s and hadronization scale, with the latter placed at $p_{\perp\mathrm{had}} = 0.5\,\mathrm{GeV}$.

We note that by taking $C_{00} = 8/3$ and all other coefficients equal to zero, the three-gluon matching term is zero. This means that the complete $H \to ggg$ amplitude has been absorbed into the Sudakov factor.

5.8 Preliminary Results

We now turn to a comparison between results obtained using a few different parameter and variable choices with the VINCIA code (using the VINCIA plug-in module with PYTHIA8). Recall that we are here studying pure gluon evolution in the fictitious decay of a scalar to two gluons. We thus intend these results mostly for illustration of the method. We set the scalar mass to $120\,\mathrm{GeV}$ and use a 1-loop running α_s with $\alpha_s(m_Z) = 0.120$. The hadronization scale is chosen to be $Q_\mathrm{I} = 1\,\mathrm{GeV}$, cf. eq.(36), corresponding to $p_{\perp\mathrm{had}} = 0.5\,\mathrm{GeV}$. In all cases we have switched off hadronization so as not to uninentionally obscure the differences between the partonic evolutions.

As a first sanity check, we compare VINCIA to PYTHIA8. To make the comparison as direct as possible, we make the following non-default choices in PYTHIA8: an unpolarized pure-gluon evolution is obtained by turning off gluon polarization effects as well as $g \to q\bar{q}$ splittings, and we use $\alpha_s(m_Z) =$

0.120, as in VINCIA. The following relevant defaults were not changed: 1-loop running of α_s, with the Lund definition of $p_\perp$, $p_{\perp\mathrm{evol}}$ [3], as renormalization scale (the closest we can get to the ARIADNE $p_\perp$ that is used by default in VINCIA), and the hadronization cutoff $p_{\perp\mathrm{evol}} = 0.5\,\mathrm{GeV}$, again as similar as possible to ours, though not identical. The plot in fig. 3 shows the 3-, 4-, and 5-jet inclusive fractions as a function of the logarithm of Durham k_T (using the default PYTHIA8 Durham clustering algorithm). Results are shown for PYTHIA8 (solid) and VINCIA with two different choices of the evolution variable: $p_\perp$-ordering (dashed) and mass-ordering (dot-dashed). For both VINCIA evolution variables, the parton-shower-like kinematics map, ψ_{PS}, was used, and $C_{00} = 8/3$, i.e., the shower should correctly reproduce the tree-level 3-gluon matrix element.

We draw two conclusions from fig. 3. Firstly, even with the choices defined above, VINCIA and PYTHIA should still differ by finite terms in the radiation functions, by the definitions of transverse momentum and evolution variable, and by the kinematics maps. The fact that large differences between PYTHIA and VINCIA are absent in the jet rates is thus more than just a nice cross-check on our implementation, it also shows that the standard PYTHIA gluon evolution closely reproduces the $H \to ggg$ matrix element, as contained in the VINCIA shower. Secondly, the absence of large differences between the two different VINCIA showers underscores one of the main differences between conventional parton showers and dipole-antenna showers (see also [15]); in the dipole-antenna formulation, the radiation patterns are intrinsically generated by coherent pairs of partons in a Lorentz-invariant way, and hence the evolution variable no longer affects this behaviour to first order. The remaining genuine ambiguity associated with the choice of evolution variable is a small effect.

Next, we turn to variations of the VINCIA shower itself. The plot in fig. 4 illustrates the distribution of the (symmetrized) type-I resolution scale for three-parton configurations, obtained with unmatched VINCIA for "soft" (all $C_{\alpha\beta} = 0$), "standard" ($C_{00} = 2.66 \sim 8/3$) and "hard" ($C_{00} = 10$) variants. For all curves, we use $\mu_R = Q_{\mathrm{I}}/2 \sim p_\perp$. The point $Q = 80\,\mathrm{GeV}$ corresponds to the "Mercedes" configuration. While the variations greatly affect the shape of the distribution, the peak position remains fairly stable, here at around a tenth of the original mass.

To investigate how matching reduces this uncertainty, fig. 5 shows the two-dimensional phase space population for three-parton configurations corresponding to five different settings of the VINCIA plug-in, from top left to bottom right: soft (unmatched), soft (matched), standard, hard (unmatched), hard (matched), where matched here refers to matching to $H \to ggg$ at tree level. As in fig. 2, the dipole phase space is represented as a triangle in the two phase space invariants, $y_{ij} = s_{ij}/s$ and $y_{jk} = s_{jk}/s$, here symmetrized over i, j, and k (because gluons are indistinguishable). The dark color in the center of the plots indicates low probability, with warmer colors (lighter shades) denoting increasing probabilities towards the corners and sides. In order to focus on the hard central region, the color scale has been forced to white for $1/N dN/dy_{ij}dy_{jk} \geq 2$, thus "whiting out" the strong peaking towards the corners and sides of the triangle which would otherwise dominate. In the top row, corresponding to the soft shower, the matching fills in missing configurations in the near-"Mercedes" region. In the middle row, no matched plot is shown, because setting $C_{00} = 8/3$ corresponds to exponentiating the matrix element itself and hence the shower already produces the "correct" result in this case. For the lower two plots, the shower is significantly harder than the matrix element. In this case, the code responds by generating a matching term with negative weight, effectively reducing the population of the hard region when "added" to the unmatched events. The reduction in uncertainty for this observable is evident by comparing the variation from top to bottom on the left, with the variation on the right. (The somewhat odd-looking contours are merely an artifact of ROOT's contour algorithm operating on binned histograms.) We note that the

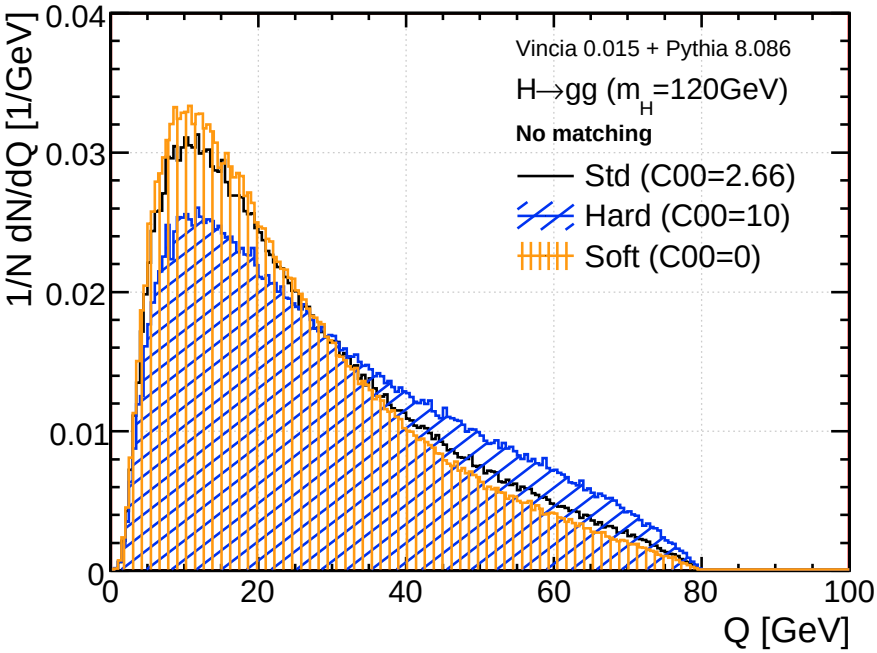

Figure 4: Type I evolution scale for the first shower branching, symmetrized over a, r, and b: $Q = \min(Q_{\rm I}(a,r,b), Q_{\rm I}(r,b,a), Q_{\rm I}(b,a,r)) \sim 2p_\perp$, for $H \to gg$ with $m_H = 120\,$GeV, showered with three different choices of the finite terms in the VINCIA shower, corresponding to "standard" ($C_{00} = 2.66$), "hard" ($C_{00} = 10$), and "soft" ($C_{00} = 0$) variations.

dependence on genuine higher-order corrections of course remains. We have here varied only the finite terms in the shower, in order to focus on that particular source; the distributions shown here should not be interpreted as indicative of the entire remaining uncertainty.

In fig. 6 we show the number of two-jet configurations as a function of the type-I resolution scale, roughly equivalent to the Sudakov factor expressed in this variable, before (left) and after (right) matching to the tree-level $H \to ggg$ matrix element. As could be expected, the uncertainty on this observable is greatly reduced by matching to this level.

The thrust is a more complicated observable, whose distribution is shown in fig. 7, again with (right) and without (left) matching. We see that, in the region accessible to 3-parton configurations, $1 - T < 1/3$, the variation is indeed canceled, while in the region of large $1 - T$, accessible only to 4-parton and higher configurations, some uncertainty remains (though with 2.5M events generated per curve, small statistical fluctuations also become noticeable in the sharply-dropping tail). We also note that the behavior at small $(1 - T)$ is very sensitive to the choice of μ_R, a point to which we plan to return in the future. We have used $\mu_R = Q_{\rm II}/2$ here.

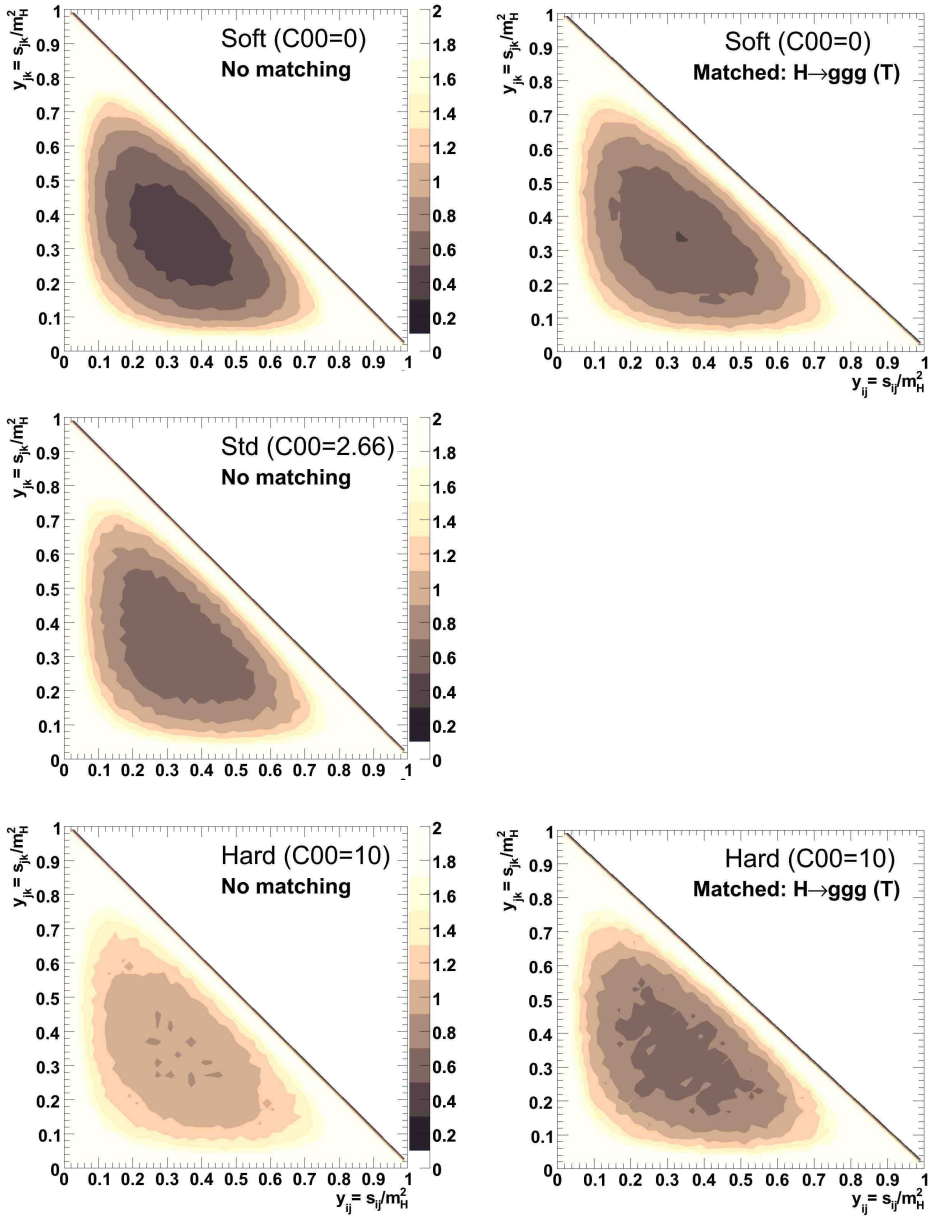

Figure 5: Phase space population $1/N dN/dy_{ij}/dy_{jk}$, for three-gluon configurations, symmetrized over all combinations of i, j, and k. Top left: the "soft" VINCIA shower off two-parton configurations, with all $C_{ij} = 0$. Top right: as in top left, but including the tree-level $H \to ggg$ matching term. Note how the radiation "hole" in the center is filled in slightly. Middle: the "standard" VINCIA shower, which absorbs the tree-level $H \to ggg$ matrix element correction. Lower left: same as upper left, but with $C_{00} = 10$, corresponding to a "hard" VINCIA shower. Lower right: as in the lower left, but including tree-level matching. In this case, the $H \to ggg$ correction is negative, reducing the population of the central region.

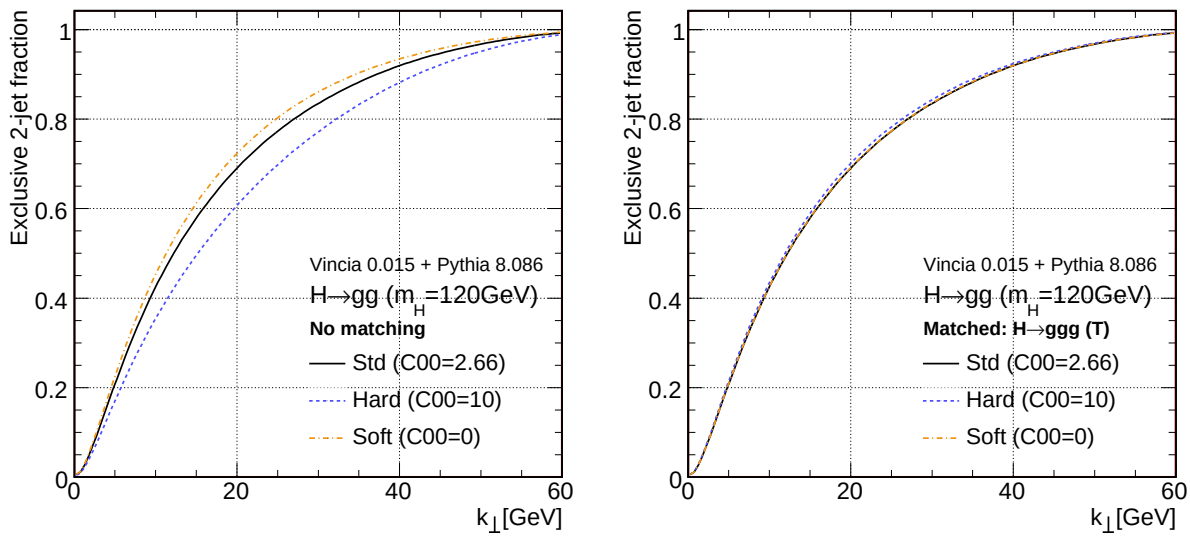

Figure 6: The number of exclusive two-jet configurations as a function of DURHAM $k_\perp$. Labels are identical to those in fig. 4, except that we have here used a different renormalization scale choice, $\mu_R = Q_{\rm II}/2$, translating to $\mu_R = m_H/2$ for the first branching over all of phase space. The kinematics map should not lead to big effects and was set to $\psi_{\rm ARIADNE}$ for all plots.

6 Still Deeper?

In sect. 3, we discussed how to match a leading-logarithmic parton shower to tree-level calculations with an arbitrary preselected number of resolved partons. We also outlined how to perform a similar matching at one loop. In the previous section, we presented a first implementation of these ideas. While there is still a great deal of work to be done in fleshing out and implementing our approach, it is also interesting to peer ahead, and ask: how would one go further in perturbation theory? How could one further improve the accuracy of parton-shower predictions?

One can presumably proceed to higher fixed orders, matching to NNLO calculations by deriving generalizations of the equations presented in sect. 3. We shall not examine such matching in any greater detail. Instead, let us explore the possibility of resumming subleading logarithms, that is including not only terms of $\mathcal{O}(\alpha_s^n \ln^{2n,2n-1} y)$ where y is a large ratio of scales, but down to $\mathcal{O}(\alpha_s^{n+1} \ln^{2n,2n-1} y)$.

For this purpose, it is crucial to have a formalism that treats all the leading-logarithmic singularities exactly point-by-point in phase space. This is true of the antenna-based formalism described here (and would also be true of showering based on the Catani–Seymour dipole formalism). To set up a subleading-logarithmic shower, we must consider corrections to the showering kernel itself. There are two kinds of corrections: virtual corrections, and real-emission ones. The former still correspond to a $2 \to 3$ branching process, but with the branching probability computed to one order beyond leading in α_s. The real-emission corrections correspond to a new branching process, $2 \to 4$ partons. Such a branching can occur with the basic Sudakov, of course, but only in two branching steps. Here, it will sometimes happen in one step. Indeed, the kernel that will ultimately enter a modified Sudakov factor

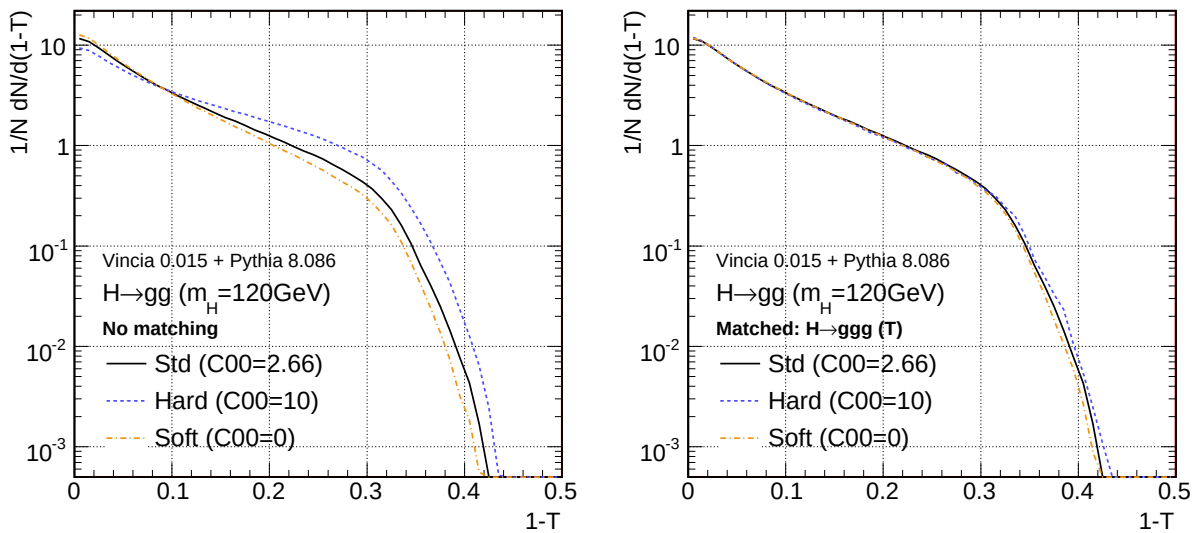

Figure 7: Differential Thrust $(1-T)$ distribution. Labels are identical to those in fig. 4.

will not be simply the $2 \to 4$ branching or antenna function, but rather that function, with the iterated $2 \to 3$ contribution subtracted out. This excess represents the genuine *correlated* $2 \to 4$ branching probability. The required ingredients in a dipole-antenna approach — 1-loop corrections to the basic antenna, and the $2 \to 4$ tree-level antenna function — are known from the development of an NNLO fixed-order formalism [40].

We would further need a definition of the evolution variable that can be evaluated on $n \to n+2$ branchings, and that regulates all infrared divergences in them. We would also need an appropriate phase-space mapping for the following factorization,

$$\frac{\mathrm{d}\Phi_{n+2}^{[i]}}{\mathrm{d}\Phi_n} = \frac{\mathrm{d}\Phi_{n-2}^{[i]}}{\mathrm{d}\Phi_{n-2}} \frac{\mathrm{d}\Phi_4^{[i]}}{\mathrm{d}\Phi_2} \,, \tag{62}$$

which is now six- rather than three-dimensional. With these definitions and mappings, the NLL Sudakov would presumably take the form,

$$\begin{aligned} \Delta_{\mathrm{NLL}}(t_{\mathrm{n}}, t_{\mathrm{end}}) \;&=\; \Delta'_{\mathrm{LL}}(t_{\mathrm{n}}, t_{\mathrm{end}}) \times \\ &\exp\left[-\int_{t_{\mathrm{in}}}^{t_{\mathrm{end}}} \mathrm{d}t_{n+2} \sum_{j\in\{n\to n+2\}} \int \frac{\mathrm{d}\Phi_{n+2}^{[j]}}{\mathrm{d}\Phi_n}\, \delta(t_{n+2} - t^{[j]}(\{p\}_{n+2}))\, A_{2\to 4}^{[j]}(\{p\}_n \to \{p\}_{n+2}) \right] , \end{aligned} \tag{63}$$

where Δ'_{LL} includes the 1-loop corrections to the LL kernel. We will face the issue of fixing the finite terms in the LL Sudakov factor upon matching to the NLL one, and a related issue of maintaining the positivity of the resulting NLL corrections $A_{2\to 4}$. The requirement of maintained positivity is crucial to a probabilistic interpretation, such as the Markov chain.

Assuming these issues can be resolved in a satisfactory manner, the matching prescriptions appear to generalize in a straightforward way. At order α_s, nothing much changes, except for possible mod-

ifications to the LL kernels. At relative order α_s^2, the shower now simply produces one extra term, corresponding to a direct $2 \to 4$ branching. It contains the proper subleading logarithms by construction, and accordingly the tree-level matching equation will have the form (differentially in $\mathrm{d}\Phi_{X+2}$),

$$\begin{aligned} w_{X+2} &= |M_{X+2}|^2 - \sum_i A^{[i]}_{2\to 3}(...)|M_{X+1}|^2\Theta(t^{[i]}(\{p\}_{X+2}) - t_{X+1}) \\ &\quad - \sum_j A^{[j]}_{2\to 4}(...)|M_X|^2\Theta(t^{[j]}(\{p\}_{X+2}) - t_X)\ ; \quad t < t_{\mathrm{had}}\ , \end{aligned} \tag{64}$$

where $A^{[i]}_{2\to 3}$ includes 1-loop corrections and $A^{[j]}_{2\to 4}$ is the direct $2 \to 4$ kernel, as discussed above.

At least from this perspective, the inclusion of explicit $n \to n+2$ branchings poses no fundamental problem. In addition, inclusion of matching to 1-loop and 2-loop matrix elements, which will both be modified by subtraction terms, would open the way to event generation at NNLO.

7 Conclusion and Outlook

We have presented a new general formalism for parton-shower resummations. The formalism allows us to explore both the uncertainties inherent in the parton-shower predictions, and the reductions in them possible by matching to fixed-order matrix elements. We keep track of the ambiguities of the shower approach away from the soft and collinear regions, allowing us gain a systematic estimate of the associated uncertainties. The quantification of these uncertainties, as well of their reduction by matching, is novel.

We have outlined a general approach for matching to fixed-order matrix elements, based on a subtraction approach which generalizes that of Frixione and Webber. We also presented a specific algorithm based on antenna factorization and dipole-antenna showers, generalizing that of Gustafson and Lönnblad. The formalism is simple and intuitive, but is powerful enough to match fixed-order matrix elements at higher multiplicity both at tree- and 1-loop level. In this respect, it provides a generalization of both the CKKW and MC@NLO approaches. The (arbitrary) choice of non-singular terms in the shower kernel is explicitly canceled by the matching terms, which allows us to quantify the degree to which matching to a given order reduces the uncertainty inherent in parton-shower predictions.

We presented a generalization of the definition of the hadronization cut-off that would make possible a more universal modeling of non-perturbative physics, allowing more meaningful comparisons of different parton-shower approaches, as well as the improvement of fixed-order parton-level calculations without reference to a specific hadronization model.

We have developed a proof-of-concept level implementation for matching of gluon showers in the decay process $H \to gg$ including both real and virtual corrections, in the form of the VINCIA code, and have presented illustrative comparisons with and without matching for a few benchmark distributions.

The next step will be to include quarks and perform a more comprehensive study of both $H \to gg$ and $Z \to q\bar{q}$ fragmentation, exploring the properties of the VINCIA algorithm and its relation to existing approaches in greater detail. We plan to go into greater detail on various theoretical aspects in a future paper [31]. We believe that it should be straightforward to automate matching for general lepton collider and decay processes matrix elements, once the evolution variables are generalized to be history-independent. The inclusion of initial-state radiation and matching will be necessary to extend the approach to hadron collisions. The formalism outlined here should be sufficiently general to make

this feasible. Indeed, as we have discussed briefly, we believe it will be sufficiently general to open a path to matching and showering deeper into the perturbative regime, both in powers of α_s and orders of subleading logarithms.

Acknowledgments

We thank Z. Bern, R. Frederix, A. Gehrmann–De Ridder, S. Mrenna and T. Sjöstrand for help and valuable comments. W. G. and P. S. are supported by Fermi Research Alliance, LLC, under Contract No. DE-AC02-07CH11359 with the United States Department of Energy. D. A. K. is supported in part by the Agence Nationale de la Recherche of France under grant ANR-05-BLAN-0073-01. The *Service de Physique Théorique* is a laboratory of the *Direction des Sciences de la Matière* of the *Commissariat à l'Energie Atomique* of France.

References

[1] M. Bengtsson and T. Sjöstrand, Phys. Lett. B **185**, 435 (1987);
M. Bengtsson and T. Sjöstrand, Nucl. Phys. B **289**, 810 (1987);
G. Miu and T. Sjöstrand, Phys. Lett. B **449**, 313 (1999) [hep-ph/9812455];
E. Norrbin and T. Sjöstrand, Nucl. Phys. B **603**, 297 (2001) [hep-ph/0010012].

[2] T. Sjöstrand, S. Mrenna and P. Skands, JHEP **0605**, 026 (2006) [hep-ph/0603175].

[3] T. Sjöstrand and P. Z. Skands, Eur. Phys. J. C **39**, 129 (2005) [hep-ph/0408302].

[4] G. Marchesini and B. R. Webber, Nucl. Phys. B **238**, 1 (1984).

[5] G. Corcella *et al.*, JHEP **0101**, 010 (2001) [hep-ph/0011363].

[6] M. H. Seymour, Nucl. Phys. B **436**, 443 (1995) [hep-ph/9410244];
M. H. Seymour, Comput. Phys. Commun. **90**, 95 (1995) [hep-ph/9410414].

[7] S. Catani, F. Krauss, R. Kuhn and B. R. Webber, JHEP **0111**, 063 (2001) [hep-ph/0109231].

[8] J. André and T. Sjöstrand, Phys. Rev. D **57**, 5767 (1998) [hep-ph/9708390];
S. Mrenna, hep-ph/9902471.

[9] S. Höche, F. Krauss, N. Lavesson, L. Lönnblad, M. Mangano, A. Schälicke and S. Schumann, hep-ph/0602031.

[10] T. Gleisberg, S. Höche, F. Krauss, A. Schälicke, S. Schumann and J.-C. Winter, JHEP **0402**, 056 (2004) [hep-ph/0311263].

[11] M. L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A. D. Polosa, JHEP **0307**, 001 (2003) [hep-ph/0206293].

[12] E. Boos *et al.*, hep-ph/0109068;
J. Alwall *et al.*, Comput. Phys. Commun. **176**, 300 (2007) [hep-ph/0609017].

[13] L. Lönnblad, JHEP **0205**, 046 (2002) [hep-ph/0112284].

[14] N. Lavesson and L. Lönnblad, JHEP **0507**, 054 (2005) [hep-ph/0503293].

[15] G. Gustafson, Phys. Lett. B **175**, 453 (1986).

[16] G. Gustafson and U. Pettersson, Nucl. Phys. B **306**, 746 (1988).

[17] L. Lönnblad, Comput. Phys. Commun. **71**, 15 (1992).

[18] S. Mrenna and P. Richardson, JHEP **0405**, 040 (2004) [hep-ph/0312274].

[19] F. Maltoni and T. Stelzer, JHEP **0302**, 027 (2003) [hep-ph/0208156].

[20] S. Frixione and B. R. Webber, JHEP **0206**, 029 (2002) [hep-ph/0204244].

[21] S. Frixione and B. R. Webber, hep-ph/0612272.

[22] P. Nason, JHEP **0411**, 040 (2004) [hep-ph/0409146];
M. Krämer, S. Mrenna and D. E. Soper, Phys. Rev. D **73**, 014022 (2006) [hep-ph/0509127];
P. Nason and G. Ridolfi, JHEP **0608**, 077 (2006) [hep-ph/0606275].

[23] Z. Nagy and D. E. Soper, JHEP **0510**, 024 (2005) [hep-ph/0503053];
Z. Nagy and D. E. Soper, hep-ph/0601021.

[24] Z. Nagy and D. E. Soper, arXiv:0706.0017 [hep-ph].

[25] C. R. Schmidt, Phys. Rev. Lett. **78**, 4531 (1997) [hep-ph/9612454];
H. Jung and G. P. Salam, Eur. Phys. J. C **19**, 351 (2001) [hep-ph/0012143];
H. Jung, Comput. Phys. Commun. **143**, 100 (2002) [hep-ph/0109102];
G. Gustafson, L. Lönnblad and G. Miu, Phys. Rev. D **67**, 034020 (2003) [hep-ph/0209186];
E. Avsar, G. Gustafson and L. Lönnblad, JHEP **0701**, 012 (2007) [hep-ph/0610157];
K. Golec-Biernat, S. Jadach, W. Płaczek, P. Stephens and M. Skrzypek, hep-ph/0703317.

[26] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **65**, 054022 (2002) [hep-ph/0109045].

[27] C. W. Bauer and M. D. Schwartz, hep-ph/0607296;
C. W. Bauer and F. J. Tackmann, arXiv:0705.1719 [hep-ph].

[28] S. Frixione, P. Nason and C. Oleari, arXiv:0709.2092 [hep-ph].

[29] Y. I. Azimov, Y. L. Dokshitzer, V. A. Khoze and S. I. Troian, Phys. Lett. B **165**, 147 (1985).

[30] D. A. Kosower, Phys. Rev. D **57**, 5410 (1998) [hep-ph/9710213];
J. M. Campbell, M. A. Cullen and E. W. N. Glover, Eur. Phys. J. C **9**, 245 (1999) [hep-ph/9809429].

[31] W. T. Giele, D. A. Kosower, and P. Z. Skands, in progress.

[32] T. Sjöstrand, S. Mrenna and P. Skands, arXiv:0710.3820 [hep-ph].

[33] A. A. Markov, Izvestiia Fiz.-Matem. Obsch. Kazan Univ., (2nd Ser.), 15(94):135 (1906) [English translation in O.B.Sheynin, ed., *Probability and Statistics. Russian Papers* (NG Verlag, Berlin, 2004).];
M. K. Ng, W. K. Ching, *Markov Chains: Models, Algorithms and Applications*, (Springer, 2006; ISBN 0387293353).

[34] V. V. Sudakov, Sov. Phys. JETP **3**, 65 (1956) [Zh. Eksp. Teor. Fiz. **30**, 87 (1956)].

[35] R. Kleiss, Phys. Lett. B **180**, 400 (1986).

[36] T. Plehn, D. Rainwater and P. Skands, Phys. Lett. B **645**, 217 (2007) [hep-ph/0510144].

[37] A. Pukhov *et al.*, hep-ph/9908288;
A. Pukhov, hep-ph/0412191.

[38] S. Catani and M. H. Seymour, Phys. Lett. B **378**, 287 (1996) [hep-ph/9602277];
S. Catani and M. H. Seymour, Nucl. Phys. B **485**, 291 (1997) [Erratum-ibid. B **510**, 503 (1998)] [hep-ph/9605323].

[39] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, JHEP **0509**, 056 (2005) [hep-ph/0505111].

[40] D. A. Kosower, Phys. Rev. D **67**, 116003 (2003) [hep-ph/0212097];
D. A. Kosower, Phys. Rev. Lett. **91**, 061602 (2003) [hep-ph/0301069];
D. A. Kosower, Phys. Rev. D **71**, 045016 (2005) [hep-ph/0311272];
A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, Phys. Lett. B **612**, 36 (2005) [hep-ph/0501291];
A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, Phys. Lett. B **612**, 49 (2005) [hep-ph/0502110].