XII. Growth of Structure in the Universe

The Big Bang theory describes a universe that is perfectly homogeneous, yet the real universe is decidedly not so. An important question is whether the structure that we observe a serious problem for the Big Bang or is it inconsequential - *e.g.*, in the same way that the surface of the Earth is high structured on small scales but nearly a perfect spheroid on large scales. The answer is that structure in the universe is indeed compatible with Big Bang theory. The current thinking is that the seeds of the large structures that we observe today (galaxies, galaxy cluster, superclusters, *etc.*) began as small density perturbations (of order, say, 10^{-4} initially) that grew as a result of gravitational instability into the structures that we observe today. The question then becomes what was the origin of the perturbations, and one possibility is that they arose at the end of the inflationary epoch as described in the previous chapter. Again, the theory of how such perturbations arose is beyond the scope of this book. This chapter will focus on how such perturbations can be characterized and how they grow in time. This chapter is meant to be only an introduction to the subject - a fuller treatment may be found in Peebles *Large Scale Structure in the Universe.*

A. Nonlinear Growth for Top-Hat Perturbations

For the moment, consider a universe in which the mean density is everywhere exactly the critical density except inside a spherical region where there is a slight excess or deficit in density relative to the outside. To a first approximation, the sphere will evolve as either a miniature closed or open universe, and so relative to its surroundings, it will either stop expanding eventually and collapse (forming, say, a cluster of galaxies) or expand at a rate in excess of its surroundings and ultimately form a void, or region of low density. As will be shown later, such behavior occurs primarily during the matter-dominated era after recombination. Before that time matter is tightly couple to the radiation field, and the radiation pressure prevents the perturbation from growing.

The equations describing the growth of the perturbation can be derived as follows. First, consider an overdense perturbation during the matter dominated era after recombination. A perturbation is characterized by two parameters. Take a sphere of radius r, density ρ. Let $v = \dot{r}$ be the velocity of a particle on the surface relative to the center. The total mass is $M = (4/3)\pi r^3 \rho$. Let $\alpha = GM/r - \frac{1}{2}v^2$ be the energy per unit mass of a particle on the surface. Thus, the parameters that characterize the perturbation are the mass M and the energy parameter α. This equation can be integrated just as was done for the cosmological models in Chapter 5; the result is

$$r = \frac{GM}{2\alpha}(1 - \cos\theta) \qquad (12.1)$$

$$t = \frac{GM}{(2\alpha)^{3/2}}(\theta - \sin\theta) \tag{12.2}$$

Note that if r were the cosmological radius R, then α would have the value $\frac{1}{2}c^2$.

If the perturbation remained perfectly smooth, it would stop expanding at some point and try to recollapse back to a singularity. In practice, irregularities in it also grow, and if things are dissipationless, the matter will eventually form a virialized system.

The maximum radius is reached when $v = 0$:

$$r_{max} = \frac{GM}{\alpha}. \tag{12.3}$$

The <u>total</u> energy in the perturbation is

$$E = W = -\frac{3}{5}\frac{GM^2}{r_{max}}. \tag{12.4}$$

If the perturbation collapses and conserves energy (no radiation and no particles ejected), then the virial theorem gives

$$W_{now} = -2T_{now}. \tag{12.5}$$

But

$$E = W + T = -T_{now} = -\frac{3}{2}M\langle\sigma_{1D}^2\rangle. \tag{12.6}$$

Combining,

$$E = -\frac{3}{5}M\alpha = -\frac{3}{2}M\sigma_{1D}^2. \tag{12.7}$$

Here, σ_{1D} is the one dimensional velocity dispersion of the system today. Thus,

$$\alpha = \frac{5}{2}\sigma_{1D}^2. \tag{12.8}$$

This relates the parameter α to a quantity that is observable today.

The collapse occurs at the time when $\theta = 2\pi$:

$$t_c = \frac{2\pi GM}{(2\alpha)^{3/2}} = 0.56\frac{GM}{\sigma_{1D}^3}. \tag{12.9}$$

The behavior of the initial growth of the perturbation is of some interest. In Eqs. 12.2, it is possible to eliminate θ from the parametric equations and relate r

and t directly. It is necessary to expand the right hand sides to 4th order:

$$\begin{aligned} r =& \frac{GM}{2\alpha}\left[\frac{\theta^2}{2} - \frac{\theta^4}{24}\right] && (12.10a) \\ =& \frac{GM}{2\alpha}\frac{\theta^2}{2}\left[1 - \frac{\theta^2}{12}\right] && (12.10b) \end{aligned}$$

$$\begin{aligned} t =& \frac{GM}{(2\alpha)^{3/2}}\left[\frac{\theta^3}{6} - \frac{\theta^5}{120}\right] && (12.11a) \\ =& \frac{GM}{(2\alpha)^{3/2}}\frac{\theta^3}{6}\left[1 - \frac{\theta^2}{20}\right] && (12.11b) \end{aligned}$$

Let

$$u = \left[\frac{6t}{GM}(2\alpha)^{3/2}\right]^{1/3} = \theta\left[1 - \frac{\theta^2}{60}\right] \tag{12.12}$$

Then,

$$\theta = u\left[1 + \frac{u^2}{60}\right] \tag{12.13}$$

and finally,

$$r = \left(\frac{9GM}{2}\right)^{1/3} t^{2/3}\left[1 - \left(\frac{6t}{GM}\right)^{2/3}\frac{\alpha}{10}\right]. \tag{12.14}$$

The 1st order term is just the expansion of the universe. The second order term represents the 1st order deviation from uniform expansion.

It is often convenient to express the deviations of a perturbation relative to unperturbed Hubble flow in the form:

$$\begin{aligned} r =& r_0(1 - \epsilon) && (12.15a) \\ \rho =& \rho_0(1 + \delta). && (12.15b) \end{aligned}$$

Here, r_0 and ρ_0 are the radius and density of a sphere of unperturbed space with the same mass as the perturbed region at the same time. All quantities in Eq. (12.15) are a function of time. From conservation of mass, $M = (4/3)\pi r^3\rho$, one finds that $\delta = 3\epsilon$. By comparison with Eq. 12.14, the overdensity parameter δ is found to depend on time as

$$\delta = \frac{3}{20}\left(\frac{12\pi t}{t_c}\right)^{2/3} = 1.69\left(\frac{t}{t_c}\right)^{2/3}, \tag{12.15}$$

where the definition of collapse time (Eq. (12.9) has been used in place of α.

EXAMPLE: The Coma Cluster has a mass $M = 1.5 \times 10^{15} h^{-1}$ $M_\odot$ and a 1-D velocity dispersion of $\sigma_{1D} = 1000$ km s^{-1}. The collapse time is $t_c = 3.7 \times 10^9 h^{-1}$

yrs. At recombination, $t = 130,000/\sqrt{\Omega_0 h^2}$, giving an overdensity at that time of $\delta = 1.8 \times 10^{-3}/\Omega^{1/3}$.

The above treatment of perturbations is incomplete in that it ignores what happens after collapse. In the ideal top-hat perturbation, material will continue to infall after the collapse, because material outside the perturbation will still have a negative total energy with respect to the perturbation. The global properties of a galaxy cluster are the same whether the cluster formed from a low mass, high overdensity perturbation where collapse was followed by significant infall or from a larger, lower density perturbation that had little subsequent infall.

Next, consider an underdense perturbation. Here the situation is more complicated because shells of material at small radii now expand faster than their lesser perturbed counterparts at large radii, leading to shell crossing. Let

$$\alpha = \frac{1}{2}v^2 - (GM/r) = \text{constant} > 0 \tag{12.16}$$

be the positive energy of a particle sitting on the edge of the perturbation. Consider the perturbations growth in 2 stages:

1. Initial Growth: The equations describing the initial growth are those of an open universe:

$$r = \frac{GM}{2\alpha}\left[\cosh\theta - 1\right] \tag{12.18}$$

$$t = \frac{GM}{(2\alpha)^{3/2}}\left[\sinh\theta - \theta\right]. \tag{12.19}$$

After a time $t \approx GM/(2\alpha)^{3/2}$, the density contrast inside and outside the perturbation is of order unity. An upper limit to the void size is given by computing r for $t = t_0$.

The density contrast can be computed in many ways, but the easiest is to use

$$\Omega_V = \frac{\Omega_0(1+z)}{1+\Omega_0 z}. \tag{12.20}$$

Here, Ω_V is the density in the void at an early epoch: $\Omega_V = 1 - \delta$. Solving for Ω_0:

$$\Omega_0 = \frac{\Omega_V}{1 + z - \Omega_V z} = \frac{1}{1 + \delta z}. \tag{12.21}$$

EXAMPLE: The void counterpart to the Coma cluster has $\alpha = (5/2)\sigma^2$ where $\sigma = 1000$ km s^{-1}, $M = 1.5 \times 10^{15} h^{-1}$ M$_\odot$. This yields $\sinh\theta - \theta \approx 11$, $\theta \approx 3.36$, $r \leq 17h^{-1}$ Mpc. The density contrast (with $\delta = 2 \times 10^{-3}$ at $z = 1400$) is $\Omega_0 = 0.29$.

2. Late Growth: For unusually underdense perturbations with evacuation times small compared to the age of the universe, there is a significant shell phase - the void empties out and the material piles up in a surrounding shell. An approximate calculation of the growth of the shell can be done as follows. At some early time t_i, let δ_i be the underdensity inside the initial underdense region, α_i be the energy per unit mass of a particle on the edge of that region, and M_i be the mass inside the region. We have (Eq. 12.16)

$$\delta_i = \frac{3\alpha_i}{10}\left(\frac{6t_i}{GM_i}\right). \tag{22}$$

A particle outside that region (in unperturbed space) will still see a net underdensity of

$$\delta = \frac{\Delta M}{M} = \delta_i \frac{M_i}{M}, \tag{12.23}$$

where M is the total mass out to the radius of that particle. The mean energy per unit mass of such a particle is

$$\alpha \propto \delta M^{2/3} \propto \delta_i M^{-1/3}, \tag{12.24}$$

or

$$\alpha = \alpha_i \left(\frac{M_i}{M}\right). \tag{12.25}$$

This equation give the energy per particle at a distance r from the center [with $M = (4/3)\pi r^3 \rho_0$]. The total energy out to some radius or mass is

$$E = \int_0^r \alpha(r')4\pi r'^2 \rho_0 dr' = \frac{3}{2}\alpha_i M \left(\frac{M_i}{M}\right)^{1/3} = \frac{3}{2}\alpha_i M_i \left(\frac{M}{M_i}\right)^{2/3}. \qquad 12.26$$

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Assume that the void grows in a self-similar fashion:

$$r = r_i \left(\frac{t}{t_i}\right)^{\gamma}, \quad M \propto \rho r^3. \tag{12.27}$$

But $\rho \propto 1/t^2$, so

$$M = M_i \left(\frac{t}{t_i}\right)^{3\gamma-2}. \tag{12.28}$$

Note also that the velocity of the shell edge is $v = \dot{r} = \gamma(r/r_i)(t/t_i)^{\gamma-1}$. Then

$$\begin{aligned} E =& \frac{3}{2}\alpha_i M_i \left(\frac{t}{t_i}\right)^{(2/3)(3\gamma-2)} \\ =& -\frac{GM^2}{2r} + \frac{Mv^2}{2} \\ =& -\frac{GM_i^2}{r_i}\left(\frac{t}{t_i}\right)^{5\gamma-4} + \frac{1}{2}M_i\gamma^2\left(\frac{r_i}{t_i}\right)^2\left(\frac{t}{t_i}\right)^{5\gamma-4}. \end{aligned} \qquad 12.29$$

This equation is meaningful only if $2\gamma - (4/3) = 5\gamma - 4$, or $\gamma = 8/9$. Now, $r_i \approx GM_i/(2\alpha_i), t_i \approx GM_i/(2\alpha_i)^{3/2}$. Combining,

$$r \approx (2\alpha_i)^{1/3}(GM_i)^{1/9}t^{8/9}. \qquad 12.30$$

This finally gives the size of a void after a long period of time given the parameters of the initial underdense region. More detailed calculation (Fillmore and Goldreich, sometime??) suggest that a factor 1.6 belongs in front.

We are now in a position to calculate the size of the void counterpart of the Coma cluster. The Coma cluster presumably consists of some initial dense region (with parameter α_i and M_i) plus accumulated infall over time. If M is the total mass now and M_i was the initial mass, then the total energy now is

$$E = \frac{3}{2}\alpha_i M_i \left(\frac{M}{M_i}\right)^{1/3}. \qquad (12.31)$$

Thus, $\alpha_i M_i^{1/3} = 2E/(3M^{2/3}) = \sigma^2 M^{1/3}$. In the void counterpart, if the corresponding shell were just now being gobbled up, then

$$r_V \approx 2G^{1/9}(\sigma^2 M^{1/3})^{1/3}t^{8/9}. \qquad (12.32)$$

Now $M = 1.5 \times 10^{15} h^{-1}$ $M_\odot$, $\sigma = 1000$ km s^{-1}, $t = (2/3)H^{-1}$. Thus, $r_V = 13h^{-1}$ Mpc which correspond to about 1300 km s^{-1} in velocity space.

All of the above discussion assumes that the unperturbed universe is critically bound $\Omega_0 = 1$. What if the universe is, in fact, open? Today, only a small number of density peaks will still be of just the right density to still be in a state of collapse. Virtually all above-critical regions of the universe will have already collapsed, and all low density regions will be exanding without change. If we define a critical redshift $z_c \approx 1/\Omega_0$, then all growth will have occured before that redshift, and all structure that we observe today will have been frozen at z_c.

B. Linear Growth In The Presence Of Dark Energy

The previous discussion was limited to the growth of perturbations during the matter-dominated phase of the universe. (It also was limited to growing, not decaying perturbations.) Once the universe becomes dark-energy dominated, however, the growth slows down. Indeed, tracing the growth of structure as a function of redshift is one way to probe the history of dark energy in the universe.

Here, we shall derive the equation for the growth of structure of matter perturbations in a mixed matter + dark energy universe. Unlike the case of a matter-only universe, it is not possible to solve the equations analytically into the nonlinear regime. For simplicity, we will only consider dark energy in the form of a cosmological constant ($w = -1$). We will also ignore perturbations in the dark energy density itself.

We start once again with the equations for unperturbed growth in the absence of pressure:

$$\ddot{r} = -\frac{GM(r)}{r^2}, \tag{12.33}$$

where $M(r) = M_{DM} + 4/3\pi\rho_\Lambda r^3$. Here, M_{DM},the enclosed mass of ordinary matter, and ρ_Λ, the dark energy density, are both constant.

A perturbed system is one that has the same interior mass and dark energy density but has a radius given by $r' = r(1-\epsilon)$. We have $\dot{r'} = \dot{r} - \dot{r}\epsilon - r\dot{\epsilon}$, $\ddot{r'} = \ddot{r} - \ddot{r}\epsilon - 2\dot{r}\dot{\epsilon} - r\ddot{\epsilon}$. Substituting and dividing by $1-\epsilon$:

$$\ddot{r} - 2\dot{r}\dot{\epsilon} - r\ddot{\epsilon} = -\frac{GM_{DM}}{r^2}(1+3\epsilon) + \frac{4}{3}\pi G\rho_\Lambda r. \tag{12.34}$$

Subtracting the unperturbed equation and dividing by r finally gives:

$$\ddot{\epsilon} + 2H\dot{\epsilon} = \frac{3GM_{DM}}{r^2}\epsilon = 4\pi G\rho_{DM}\epsilon. \tag{12.35}$$

This equation is the final result. Note that 3ϵ is also the fractional overdensity δ inside the sphere, so this equation applies to δ as well. It is conventional to call δ the "growth factor".