III. Metrics

In special relativity, the proper distance of an infinitesimal interval ds separating two nearby events is given by $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$, where now the full 3-dimensional space axes are included. This is one example of a metric - a way to measure distances that is invariant with respect to which inertial frame one choses. The individual dx, *etc.* will be different depending on the coordinate system used to measure them, but ds is always the same. We wish to generalize the concept of metrics to arbitrary coordinate systems.

Suppose we fill all of space with a random coordinate system. Label the axes as x_i with i running from 1 to 4. Assume that we can measure proper lengths along each axis. The coordinates themselves need not be proper lengths (*i.e.*, they can be angles, *etc.*). Let L be the cumulative proper length along any particular space-like axis or proper time along the time-like axis. Define coordinate bases l_i as follows:

$$\vec{l}_i = \frac{\partial L}{\partial x_i}\hat{x}_i. \tag{3.1}$$

If we have 2 nearby points P_1 and P_2, let them be separated by Δx_i. The separation vector between the two point is given by $\Delta\vec{l} = \sum \vec{l}_i \Delta x_i$. We define the proper distance between the two points to be $(\Delta l)^2 = \vec{l}\cdot\vec{l} = \sum \vec{l}_i \cdot \vec{l}_j \Delta x_i \Delta x_j$. Metric coefficients are defined as $g_{ij} = \vec{l}_i \cdot \vec{l}_j$. We write $ds^2 = g_{ij}dx_i dx_j$ with summation implicitly performed over i and j. g_{ij} is a tensor, but we won't ever need to worry about its tensorial character here.

If we have a curve of finite length, let s be the proper distance along it. Parameterize the curve by letting x_i be a function of s. Then

$$ds = \sqrt{g_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}}ds, \tag{3.2a}$$

and

$$s = \int \sqrt{g_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}}ds, \tag{3.2b}$$

with the integral being carried out along the path.

In 4 dimensional space-time, one can have both space-like and time-like paths. For the latter, $-ds^2$ is replaced by $d\tau^2$, but otherwise the mathematical procedures involving the metrics are identical.

Metric spaces are those spaces where lengths are definable. In three-dimensional space, we can *always* find a coordinate system that is local Cartesian; in four-dimensional space-time, the analogous local coordinate system is called Lorentzian.

If we confine ourselves to Cartesian (Euclidean) space or Lorentzian (the four-dimensional equivalent of Euclidean) spacetime, then the machinery of metrics is somewhat superfluous. However, when dealing with non-Euclidean spaces, metrics provide a powerful way of describing the geometry of those spaces.

First, consider examples of metrics in three dimensions:

$$\text{a) Cartesian } (x, y, z): \qquad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3a}$$

$$\text{b) Cylindrical } (R, \phi, z): \qquad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3b}$$

$$\text{c) Spherical } (r, \theta, \phi): \qquad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}. \tag{3.3c}$$

Thus, in spherical coordinates, any length due to motion in one coordinate has the form $\Delta s = \Delta r$, $\Delta s = r\Delta\theta$, or $\Delta s = r \sin\theta \Delta\phi$.

These are all flat (Euclidean) metrics.

Next, consider a non-Euclidean space. A simple example is the surface of a sphere. Consider a sphere with radius r_0. Introduce polar coordinates θ and ϕ as for spherical coordinates. Then the metric for distances measured along the surface of the sphere is

$$ds^2 = r_0^2 d\theta^2 + r_0^2 \sin^2\theta d\phi^2. \tag{3.4}$$

This is just part of the metric for three-dimensional space.

The geometry of the surface is intrinsically different from a plane. How can we tell? Probably the most straightforward experiment one could do is to measure the separation between two nearby nearly parallel geodesics. For concreteness, imagine two ships next to each other, each sailing southwards along a line of constant longitude (such lines being geodesics on a spherical surface). Imagine also that the ships' speeds are adjusted so that they have the same latitude at any given time. The distance D between the two ships at any time is $D = r \sin\theta\delta\phi$, where r is the earth's radius, θ is the colatitude at any time, and $\delta\phi$ is the longitude separation, here assumed to be constant. The separation between the two ships varies with the distance s that they travel:

$$\begin{aligned} \frac{dD}{ds} &= \cos\theta\delta\phi \\ \frac{d^2D}{ds^2} &= -\sin\theta\delta\phi = -\frac{D}{r_0^2}. \end{aligned} \tag{3.5}$$

We can eliminate all details of the experiment by writing

$$\frac{1}{D}\frac{d^2D}{ds^2} = -\frac{1}{r_0^2}. \tag{3.6}$$

Hence the second derivative of the separation distance is 0 if we are traveling on the surface of a plane, but non-zero if we are on the surface of a sphere. Furthermore, the magnitude of the second derivative is a measure of the curvature of the surface. On a sphere, the curvature is constant everywhere. On an ellipsoid or other more arbitrary surface, the curvature varies with position. Although there are ways of defining a unique measure of curvature at any point for those surfaces, we need not be concerned with them here. Furthemore, one can derive the radius of curvature from the metric coefficients that describe any random coordinate system, but again, we will not need that here. However, there is one interesting point to note. Suppose we were to repeat the experiment at the same place on the earth but with lines of longitude drawn relative to some other "pole" (*e.g.*, the magnetic north pole). Then the ships would travel in a different direction, start with a different separation and a different initial divergence of their trajectories. There is no combination of D and D' that provides a measure of curvature that can be expressed in a manner that is independent of the coordinate system (*i.e.*, pole) that is chosen; one must go to the second derivative, d^2D/ds^2.

The surface of a sphere has some curious properties compared with a plane. For example, if we draw a circle of radius $r_0\theta$, the circumference is $2\pi r_0 \sin\theta$. The surface area is $A = \int 2\pi r_0 \sin\theta r_0 d\theta = 4\pi r_0^2$.

A sphere in 3 dimensions can be generalized to other dimensionalities quite easily. In fact, beginning in two dimensions, we have

$$\text{2-D} \qquad \begin{aligned} x &= r_0 \cos\phi \\ y &= r_0 \sin\phi \end{aligned} \tag{3.7a}$$

$$\text{3-D} \qquad \begin{aligned} x &= r_0 \sin\theta\cos\phi \\ y &= r_0 \sin\theta\sin\phi \\ z &= r_0 \cos\theta \end{aligned} \tag{3.7b}$$

$$\text{4-D} \qquad \begin{aligned} x =& R_0 \sin r \sin\theta \cos\phi \\ y =& R_0 \sin r \sin\theta \sin\phi \\ z =& R_0 \sin r \cos\theta \\ w =& R_0 \cos r. \end{aligned} \tag{3.7c}$$

By analogy with a 3 dimensional sphere having a 2 dimensional surface, we can think of a 4 dimensional "sphere" having a 3 dimensional "surface" given by

$$x^2 + y^2 + z^2 + w^2 = R_0^2 = \text{constant}. \tag{3.8}$$

The metric for this 3 dimensional surface is

$$ds^2 = R_0^2\Big[dr^2 + \sin^2 r\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)\Big]. \tag{3.9}$$

If we draw a 2-dimensional sphere about the origin within this space, then it has a surface area given by $A = \int dl_\theta dl_\phi$ where $dl_\theta = R_0 \sin r d\theta$ and $dl_\phi = R_0 \sin r d\phi$. Then $A = 4\pi R_0^2 \sin^2 r$. The total volume of the 3 dimensional space is $V = \int A R_0 dr = 4\pi R_0^3 \int \sin^2 r dr = 2\pi^2 R_0^3$. The space is clearly quite non-Euclidean in its geometry.

This 3 dimensional space turns out to be the geometry of a closed universe, as we shall show later. Although its geometry may seem a bit peculiar, virtually all the important ramifications of the geometry on such things as how we make observations of distant objects can be inferred by analogy with the making of observations on the 2 dimensional surface of a 3 dimensional sphere.

So far, non-Euclidean geometries have been introduced by examining the surfaces of n-spheres that are embedded in Cartesian spaces of one higher dimension (*i.e.*, the 2 dimensional surface of a sphere is embedded in Cartesian 3 dimensional space). However, it is important to realize that although the concept of embedding a space in one of higher dimension may help in conceptualizing the properites of that space, the extra dimension need not have any physical significance. Indeed, it is possible to invent non-Euclidean surfaces that cannot be embedded in a Euclidean space of one higher dimension, *e.g.*, 2 dimensional surfaces with a hyperbolic geometry.

Finally, two warnings are in order regarding the discussion up to now.

1. Lines along which one of the coordinates is constant (*e.g.*, ϕ, θ) are not necessarily geodesics. To measure things like curvature, one must be sure to use geodesics.

2. Geodesics in a manfold of a higher dimensional space are not necessarily geodesics in that higher space. For example, a great circle on the surface of a sphere is a geodesic of that surface, but it is hardly a geodesic in 3 dimensional space.