V. Dynamics

The objectives of this chapter are to: 1) derive a set of parametric equations that provide the general solution for the variation of the curvature radius R as a function of cosmological time t; 2) show that these solutions fall into three categories (unbound, critically bound, and bound) and that each corresponds to its own global geometry (hyperbolic, flat, and spherical); 3) show that only two constants (H_0 and q_0) are needed to completely describe the universe today.

To specify a complete cosmological model, it is necessary to determine the most general form of the function $R(t)$. Above, we showed that $R = t$ is the solution for a freely expanding universe. Next we shall look at what happens when the universe has a finite mass density.

General Relativity (GR) incorporates four fundamental principles:

1. Gravitational and inertial masses are identical (the equivalence principle).

2. In localized volumes of space about a free-falling observer (where macroscopic velocities are small compared with the speed of light), Newtonian physics must be obeyed.

3. In localized volumes of space-time about a free-falling observer, where the mass density is 0, space-time must be Lorentzian.

4. Energy is conserved.

Principle 1 is common to Newtonian physics. Principles 2 and 3 are statements that we should recover "known physics" in domains where they have been tested and are known to be valid. Principle 2 means that in small volumes of space about a free-falling observer, one can define a gravitational potential that satisfies $\nabla^2\Phi = 4\pi G\rho$. The free-falling observer must have an instantaneous velocity that is small compared to the bulk motion of any mass. Such an observer can set up a series of experiments and correctly predict their outcome based solely on Newtonian physics. Newtonian physics also has the implied assumption that the gravitational constant G is independent of space and time, and in particular, it is constant even for observers falling in a strong gravitational field. This assumption is one form of what is called the "strong principle of equivalence".

One extra feature that is introduced by GR is if the pressure P is not negligible, the density ρ is replaced by $\rho + 3P/c^2$. For the moment we will take $P = 0$, which is nearly the case for the universe today if it is dominated by ordinary or dark matter. In Chapter 8, the case of nonzero pressure will be discussed, and even there, it will be shown that one can incorporate such situations into a generalized Newtonian framework. In chapter 11 we will reconsider a universe that is a combination of the

two situations.

Virtually all the important physics can be derived by considering a very simple system. Take a small spherical region of the universe whose surface is expanding along with the rest of the universe (*i.e.*, its surface is covered with a set of F.O.'s). Let $r(t)$ be the radius of the sphere and M be the mass inside. In Newtonian gravitational theory, because the system has spherical symmetry, the gravitational force at the surface of the sphere depends only on the total mass inside and not at all on the exterior mass. [An analogous result can be proven in general relativity: it is known as Birkhoff's theorem.]

With the above caveat, the equation for the acceleration at the surface of the sphere takes the form

$$\ddot{r} = -\frac{GM(r)}{r^2} = -\frac{4}{3}\pi r G\rho, \tag{5.1}$$

where the density $\rho = 3M/(4\pi r^3)$ has been introduced.

Next, introduce the comoving coordinate u for the surface of the sphere: $r = uR$. Remember that because u is a comoving coordinate, the surface of the sphere is given by $u =$ constant for all time. Then Eq. (5.1) becomes

$$\ddot{R} = -\frac{4}{3}\pi G R\rho, \tag{5.1a}$$

which is independent of u. Thus we have the straightforward but important result that the curvature radius R is governed by the same equation as is the radius r of an arbitrarily small sphere that obeys purely Newtonian mechanics.

The integration of Eq. (5.1a) requires that we specify an equation of state for ρ. For ordinary matter, it is sufficient to write

$$\rho R^3 = K, \qquad \rho = \frac{K}{R^3}, \tag{5.3}$$

where K is some constant. The standard technique for integrating Eq. (5.1) is to multiply by $\dot{R}$ and integrate:

$$\dot{R}\ddot{R} = \frac{1}{2}\frac{d}{dt}\left(\dot{R}^2\right) = -\frac{4}{3}GR\frac{K}{R^3}\frac{dR}{dt} = \frac{4}{3}\pi G\frac{d}{dt}\left(\rho R^2\right). \tag{5.4}$$

Hence,

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi G\rho R^2}{3} = \epsilon \quad (\text{units of cm}^2\ \text{s}^{-2}). \tag{5.5}$$

Physically ϵ is like the total energy (kinetic plus potential) of a particle, although here it would need to be the energy of a particle at $u = 1$, which is not

physically measurable in the Newtonian approximation. Equation (5.5) is known as the Lemaitre Equation.

Some obvious observations:

a) If $\epsilon > 0$, then since $\dot{R}^2 > 0$, we have $\dot{R} > 0$ always. When R is big enough, $\rho R^2 \to 0$ and $\frac{1}{2}\dot{R}^2 = \epsilon =$ constant. Such a universe expands forever and is called an open universe.
b) If $\epsilon < 0$ the following logic applies. We need $\dot{R}^2 > 0$. Since $\rho R^2 \to 0$ for large R, at some point $\dot{R}^2$ reaches 0 for an expanding universe: $\rho R^2 \approx \epsilon/G$. Hence the universe will collapse. We call such a universe closed.
c) If $\epsilon = 0$, the universe is marginally unbound.

A not so obvious observation is that ϵ cannot be specified arbitrarily. Suppose we were to integrate Eq. (5.5) once more (and we shall do so presently). Having integrated a second order differential equation, we can specify two initial conditions, *e.g.*, R and $\dot{R}$ at some fiducial time t_0. Also, as part of the initial specification of the problem, we need to specify the mass M interior to R. An entirely equivalent and more cosmologically useful set of these constants is the radius R_0, the Hubble Constant $H_0 = \dot{R}/R$, and the initial density ρ_0; in terms of this new set of constants we find that $\epsilon = \frac{1}{2}R_0^2\left(H_0^2 - 8\pi G\rho_0/3\right)$.

In the Newtonian approximation, R has no special meaning, and so two initial conditions plus mass (or density) can be specified freely. In relativity, however, R is a curvature radius that has special meaning, even though that meaning is not exhibited in the derivation so far. The connection can be made, though, by looking at Eq. (5.5) in the limit that $R \to \infty$. We pointed out that in that case, the second term (corresponding to the gravitation potential energy) is negligible, so $\frac{1}{2}\dot{R}^2 = \epsilon$. This equation corresponds to a universe that is freely expanding and where gravitational forces due to matter are negligible. But in the last chapter, we showed that in such a universe, $R = t$; hence $\dot{R} = 1$ and $\epsilon = \frac{1}{2}$. Furthermore, because ϵ is a constant, we must have $\epsilon = \frac{1}{2}$ for all time (at least in an open universe), even when gravity is not negligible. Hence, of the three constants that determine the Newtonian problem, R_0, H_0, and ρ_0, only two can be specified freely in the relativistic problem; the third is given by Eq. (5.5).

It should be kept in mind that the subscript $_0$ is applied to H, R, and ρ to designate the values that they have at some fiducial epoch t_0. In general, though, they vary with time (and hence "Hubble's Constant" is not really constant after all).

At this point it is convenient to recast some of the parameters into dimensionless form. Usually H is taken as the one constant that is dimensional. A characteristic scale density is given by $\rho_{crit} = 3H^2/8\pi G$. The boundary between open and closed universes is determined by the dimensionless constant $\Omega = \rho/\rho_{crit}$. Further

manipulation of Eq. (5.5) (with $\epsilon = 1/2$) leads to

$$R = \frac{1}{H\sqrt{1-\Omega}}. \tag{5.6}$$

What happens when $\Omega \to 1$? We have $R \to \infty$, and so we really are in the domain of Euclidean geometry on all scales. Equation (5.5) can be cast in the form

$$\frac{1}{2}\dot{r}^2 - \frac{4\pi G\rho r^2}{3} = 0, \tag{5.7}$$

where r is now the radial length between any conveniently selected set of comoving points.

What happens when $\Omega > 1$? An educated guess might be to take $\epsilon = -\frac{1}{2}$ in Eq. (5.5) and associate the solution with spherical geometry, but we would like to show so formally. The following chicanery does just that. First, start with the metric and Lemaitre equation for the open universe:

$$ds^2 = R^2\left[du^2 + \sinh^2 u d\Omega^2\right], \tag{5.8}$$

$$\frac{\dot{R}^2}{R^2} = H^2 = \frac{8\pi G\rho}{3} + \frac{2\epsilon}{R^2}. \tag{5.9}$$

Introduce new variables u and S defined by $u = iv$ and $R = -iS$. Note that $uR = vS$ and that $\sinh(iv) = i\sin v$. The metric and Lemaitre equation become

$$ds^2 = S^2\left[du^2 + \sin^2 u d\Omega^2\right], \tag{5.10}$$

$$\frac{\dot{S}^2}{S^2} = H^2 = \frac{8\pi G\rho}{3} - \frac{2\epsilon}{S^2}. \tag{5.11}$$

Thus we arrive with a set of equations where S and v can be interpreted as physically real variables. After relabelling S with R, we find

$$R = \frac{1}{H\sqrt{\Omega-1}}, \tag{5.12}$$

thus proving that $\Omega > 1$ does indeed correspond to spherical geometry.

All the above solutions are valid only for a universe with zero pressure. How valid is that assumption? There are two major sources of pressure today: galaxy random velocities and radiation pressure of the microwave background radiation. We do not have a very good measure of the rms random velocities of galaxies, but we can at least make an order of magitude estimate. For example, the Milky Way has a

motion relative to the microwave background of 600 km s^{-1}. The relevant quantities to compare are ρ and P/c^2. The latter is roughly $\rho v_{rms}^2/c^2$, so $P/c^2 \approx 5\times10^{-6}\rho$ and thus the pressure is quite negligible. Radiation pressure is given by $P/c^2 = aT^4/3c^2$ where $a = 7.16\times10^{-15}$ cgs is the radiation constant. We find $P_{rad}/c^2 = 1.4\times10^{-34}$ g cm^{-3}. Now we need the actual density of matter, which is $1.9 \times 10^{-29}\Omega h^2$. [Since the Hubble constant is not well known yet, we write all relevant quantities in terms of $h = H_0/(100 \text{ km s}^{-1}Mpc^{-1})$. Likewise, the ratio $\rho/\rho_{crit} = \Omega$ is not well known either.] Taking reasonable lower bounds on Ω (> 0.2) and h (> 0.5) gives $P_{rad}/c^2 < 1.5 \times 10^{-4}\rho$ and so it too is negligible. At earlier epochs of the universe, however, radiation pressure (and mass density) dominated, and so the above analysis did not apply then. The radiation dominated era will be examined in another chapter. There is also evidence that the universe today is dominated by a form of matter given the name "dark energy" that has negative pressure. Such a universe will be treated in more detail in Chapter 11.

An astute reader will recognize that the parameter k introduced last chapter to distinguish the different types of geometry and the parameter ϵ that distinguishes bound vs. unbound universes are related uniquely: $\epsilon = -k/2$.

General Relativity admits a more general solution to the Lemaitre equation that just (5.5). This is

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2\epsilon + \frac{\Lambda}{3}R^2, \tag{5.13}$$

where Λ is constant. Physically, Λ acts like a constant energy density $8\pi G\rho_0$, although it can be either positive or negative. Einstein felt uncomfortable with the Λ term, feeling that it had no physical basis. However, new theories of particle physics have shown that one can have matter with an equation of state that acts just like a constant density term, so cosmologies based on a nonzero Λ have been given new life. So far, such equations of state apply only to situations where the pressure is not negligible, and so the analysis in this section would not apply. For the moment, we will not consider cosmologies with nonzero Λ, but will return to them in later chapters in connection with discussions of dark energy.

DEFINITION: We define a deceleration parameter $q = -R\ddot{R}/\dot{R}^2$. From Eq. (5.1), $\ddot{R} = -4\pi G\rho R/3$, so $q = \frac{1}{2}\Omega$. This is not true for $\Lambda \neq 0$ (it is $\frac{1}{2}\Omega - \Lambda/8\pi G\rho_{crit}$).

Solution of the Equations in Time

Equation (5.5) can be written in the form

$$\dot{R}^2 = -k + \frac{8\pi G\rho R^2}{3}, \tag{5.14}$$

where now we replace ϵ in favor of k. This equation can be integrated once more to get R implicitly as a function of time. Define a characteristic mass $M_0 = \frac{4}{3}\rho\pi R^3$; this mass is proportional to the mass in a sphere with linear radius R (it is not precisely that mass because space is non-Euclidean, but the ratio of the two can be

derived easily). Furthermore, M_0 is constant with time. Then Eq. (5.14) can be written

$$\dot{R}^2 = -k + \frac{2GM_0}{R}. \tag{5.15}$$

Let $x = R/2GM_0$, $\eta = t/2GM_0$. Then

$$\left(\frac{dx}{d\eta}\right)^2 = \frac{1}{x} - k = \frac{1-kx}{x} \tag{5.16}$$

and

$$d\eta = dx\sqrt{\frac{x}{1-kx}}. \tag{5.17}$$

We can solve all cases at once by letting

$$x = S_k^2(u) = k\left[\frac{1-C_k(2u)}{2}\right]. \tag{5.18}$$

[Caution: this u is different from the u in the metric.] Then $dx = S_k C_k du$, etc. We find, after integrating,

$$\begin{aligned} \eta &= k\left[u - \frac{1}{2}S_k(2u)\right] \\ &= \frac{1}{2}k\left[\theta - S_k(\theta)\right]. \end{aligned} \tag{5.19}$$

We have taken $\eta = 0$ at $u = 0$. In terms of R and t, we have

$$R = kGM_0\left[1 - C_k(\theta)\right] \tag{5.20a}$$

$$t = kGM_0\left[\theta - S_k(\theta)\right]. \tag{5.20b}$$

Note that $dt = Rd\theta$. θ is called "conformal time".

What happens if $k = 0$? We go back to Eq. (5.3) and find that

$$\dot{R}^2 = \frac{2GM_0}{R}. \tag{5.21}$$

So

$$R = t^{2/3}\left[\frac{9GM_0}{2}\right]^{1/3}. \tag{5.22}$$

Plots of the three types of solutions are shown in Figure (5.1).

Inspection of Fig. (5.1) shows that the age of the universe is always smaller than (or equal to) $1/H$. Equality holds for an empty universe or for an open universe

in the limit of large θ. The other interesting case that can be solved exactly is for $k = 0$; combining Eqs. (5.20) with the definition of H, it is straightforward to show that $t = \frac{2}{3}H^{-1}$ at any epoch.

Summary of Constants and Variables

In the Newtonian analog of an expanding universe, there are three constants that one must specify: two constants that come from integrating the equations of motion and one constant that is the total mass in the problem. One of the integration constants can be taken to be the time of the big bang relative to some fiducial epoch, and without loss of generality that time can be taken to be $t = 0$. Hence a model is specified by two constants (e.g., total mass and total energy). Note that both have physical dimensions. In the cosmological expanding universe, we can no longer specify two dimensional constants - crudely speaking, the curvature radius R is the radius where two F.O.'s are separating at a velocity comparable to the speed of light, and this predefines one of the dimensional constants. The other constant is most naturally taken to be M_0, which defines a characteristic mass scale of the universe. The constant k, which in the Newtonian situation is unconstrained, must, in the cosmological model, be ± 1 or 0.

To describe the state of the universe at any given time since the big band requires specifying one additional constant (*e.g.*, t or θ or R). Equivalently, we can specify two of the parameters R, $\dot{R}$, or $\ddot{R}$, or two of the parameters R, H, or q (or Ω). Commonly, H and q are specified. Then some useful relations are:

$$R = \frac{1}{H\sqrt{k(2q-1)}}; \tag{5.23}$$

Differentiating Eq. (5.4) yields

$$\dot{R} = \frac{kS_k(\theta)}{1 - C_k(\theta)}; \tag{5.24}$$

Differentiating again and using the definition of q yields

$$C_k(\theta) = \frac{1}{q} - 1; \tag{5.25}$$

Using the definition of M_0 and combining with other quantities yields

$$M_0 = \frac{q}{GH\left[k(2q-1)\right]^{3/2}}; \tag{5.26}$$

and finally, Eq. (5.20) can be combined with the equation for $C_k(\theta)$ above to give the time t in terms of q and H. The equation numbered (5.25) will be quite useful later.

Evolution of the Universe with Time

The age of the universe *now* since the big bang is usually designated t_0, and all

associated parameters are designated with subscript $_0$ as well. However, it should be remembered that parameters like H are not constant for all time but vary. This variation can be found from Eqs. (5.17) and (5.19). The variation of H, q, and R with time is plotted in Fig. 5.? for the three cases:

$k = 0$: For all time, we have $R = R_0(t/t_0)^{2/3}$. Then $H = 2/3t$ and $q = 1/2$.

$k = -1$: For small t (small θ), this case behaves like a $k = 0$ universe. When $\theta \approx 1$, the universe changes over to one of free expansion: $R = t$, $H = 1/t$, $q \to GM_0/t$.

$k = +1$: Again, for small T (small θ), this case behaves like a $k = 0$ universe. When $\theta \approx 1$, the expansion slows down more rapidly than $k = 0$, $q \to \infty$, and $H \to 0$. When $\theta = \pi$, the expansion is halted, and a mirror image contraction begins. The "big crunch occurs" at a time $t = 2\pi GM_0 = 2\pi q_0/H_0\,[2q_0 - 1]^{3/2}$.

In the next chapter we shall be concerned with the observation of objects at large distances (*i.e.*, large redshift z).

Table 5.1 gives typical values of many of the parameters that have been discussed.

$H_0 = 100$, $T_H = 1/H_0 = 10^{10}$ years, $D_H = cT_H = 3 \times 10^9$ Gpc,
$M_H = c^3/H_0G = 6 \times 10^{22}$ $\mathrm{M}_\odot$

q_0	R_O	$\dot{R}_0$	θ	M_0	t_0
$+\epsilon$	D_H	1	$-\log\epsilon$	ϵM_H	T_H
0.1	$1.05D_H$	1.05	2.89	$0.14M_H$	$0.85T_H$
$0.5-\epsilon$	D_H/ϵ	$1/\sqrt{\epsilon}$	$\sqrt{8\epsilon}$	$1/\epsilon^{3/2}$	$0.67T_H$
1	D_H	1	$\pi/2$	1	$0.57T_H$