VI. The Universe as a Function of Redshift

The Robertson-Walker metric has the form

$$ds^2 = -dt^2 + R^2(t)\left[du^2 + S_k^2(u)d\Omega^2\right]. \tag{6.1}$$

(Remember that the Ω here is *not* the density parameter Ω of the previous chapter; Ω is serving double duty, but the context should always make it clear as to which is intended.) Remember that t = cosmic time, u = comoving radius, Ω is a shorthand designation for the angular coordinates θ and ϕ, and R is the linear curvature radius of the universe. S_k is either sin or sinh depending on whether the universe has spherical or hyperbolic geometry.

The goal of this chapter is to calculate the properties that an observer would measure when viewing objects at large distance. Although the R-W metric describes how we measure distances times in a surface of constant proper time, that surface is not realizable in any global laboratory (*i.e.*, Lorentz) frame except in the special case of an empty universe. Because of light travel time effects, an observer viewing a distant object will necessarily be receiving the light at a proper time that is later than when the light was emitted; in the interim, the universe has expanded. Thus, for example, the linear size of an object at large comoving distance u that subtends an angle θ is not $R_0 S_k(u)\theta$ (R_0 being the curvature radius today) but rather $R_i S_k(u)\theta$, R_i being the curvature radius at the time that the light was emitted. Observationally, we do not measure the distance to an object directly but rather its recession velocity or redshift z. Therefore, we need to relate z to both the comoving distance of an object u and the time at which the light was emitted t_i.

Redshift

The redshift z is defined in terms of the wavelength of light that is Doppler shifted due to the motion of an object. Let λ_e be the laboratory rest wavelength of some emission or absorption line in the spectrum of an object. If the object is moving relative to some observer, the wavelength measured will be Doppler-shifted to a value λ_r. The redshift is defined to be $z = (\lambda_r - \lambda_e)/\lambda_e$. In terms of frequency $\nu = c/\lambda$, we also have $z = (\nu_e - \nu_r)/\nu_r$. In principle we could also convert z to a linear velocity using the formulae of special relativity, but there will be no need to have that velocity explicitly.

Relation between R, H, and q and redshift

The first task is to relate the redshift z to the curvature radius R_i at the time that light from that object was emitted. Consider an F.O. who is viewing the universe while sitting at the coordinates $u = 0$, $t = t_0$. All light rays that it receives travel along radial, null geodesics: $ds^2 = -dt^2 + R^2 du^2 = 0$. The path travelled by a light ray is then given by

$$\frac{du}{dt} = \frac{1}{R} \qquad \text{or} \qquad u = \int_{t_i}^{t_0} \frac{dt}{R}. \tag{6.2}$$

A very useful result can be derived without integrating this equation explicitly. Look at two closely space rays that represent two successive pulses of light from a source. Let ϵ be the time interval between the two pulses as measured by the source and ϵ' be the time interval as measured by the observer. The comoving distance between the source and observer is constant for all time, so we have

$$u = \int_{t_i}^{t_0} \frac{dt}{R} = \int_{t_i+\epsilon}^{t_0+\epsilon'} \frac{dt}{R}. \tag{6.3}$$

If ϵ is small compared with $t_0 - t_i$, then to a good approximation the second integral I_2 is related to the first integral I_1 by $I_2 = I_1 + [\epsilon'/R(t_0)] - [\epsilon/R(t_0)]$. But we must have $I_1 = I_2$, so

$$\frac{\epsilon}{R(t_i)} = \frac{\epsilon'}{R(t_0)}. \tag{6.4}$$

If we are looking at an emission line from some source, then on dimensional grounds we have $\epsilon \propto (1/\nu) \propto \lambda$ and we find that

$$\frac{\epsilon}{\epsilon'} = \frac{R(t_i)}{R(t_0)} = \frac{1}{1+z}. \tag{6.5}$$

Thus we have a relation between the curvature radius at the time of emission and the redshift of an object.

The next step is to derive analogous expressions for q and H as a function of redshift. The derivation of H vs. z can proceed as follows. Start with the Lemaitre equation,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}. \tag{6.6}$$

Solve this for k:

$$k = R^2 \left[H^2 - \frac{8\pi G\rho}{3}\right]. \tag{6.7}$$

At $z = 0$, $t = t_0$, we have $R = R_0$, $H = H_0$ and $\rho_0 = 3\Omega_0 H_0^2/8\pi G$. At arbitrary z, we have $R = R_0/(1+z)$ and $\rho R^3 = \rho_0 R_0^3$ or $\rho = \rho_0(1+z)^3 = 3\Omega_0(1+z)^3 H_0^2/8\pi G$. Since k is a constant regardless of epoch, we have (from two applications of Eq. [6.7]):

$$k = R_0^2 H_0^2(\Omega_0 - 1) = R_0^2 \frac{H_0^2\Omega_0(1+z)^3 - H^2}{(1+z)^2}. \tag{6.8}$$

Solving for H, we find

$$H = H_0(1+z)\sqrt{1+\Omega_0 z}. \tag{6.9}$$

Finally, we can get q vs. z fairly easily. Start with the definition of Ω:

$$\Omega = 2q = \frac{8\pi G\rho}{3H^2}. \tag{6.10}$$

Using Eq. (6.9) for the variation of H with z and the relation $\rho = 3\Omega_0(1+z)^3 H_0^2/8\pi G$, we find that

$$q = \frac{q_0(1+z)}{1+2q_0 z}. \tag{6.11}$$

Note that in the limit $z \to \infty$, $q \to 1/2$ regardless of the value of q_0 today. Conversely, if q_0 differs even infinitesimally from 1/2 at some time, then at later times it will eventually tend to 0 or ∞. We will return to this point in a later chapter.

Relation between u and redshift

The relation between u and z is found by integrating Eq. (6.2). This can be accomplished in a straightforward fashion by use of Eq. (5.4). As was noted immediately after those equations were derived, we have $dt = Rd\theta$ (remember that θ is the "conformal time" or a dimensionless age of the universe). Then we have immediately that

$$u = \int d\theta = \theta_0 - \theta_i, \tag{6.12}$$

where the subscript $_0$ refers to now (are you listening, Yogi?) and the subscript $_i$ refers to the time that the light was emitted. The relation between θ and z follows immediately by combining Eqs. (5.22) and (6.11). We find

$$\begin{aligned} C_k(\theta_0) &= \frac{1}{q_0} - 1 \\ C_k(\theta_i) &= \frac{1}{q} - 1 = \frac{1 - q_0(1-z)}{q_0(1+z)}. \end{aligned} \tag{6.13}$$

By taking the inverse cosine of these expressions and inserting into Eq. (6.12), we get an explicit relation between z and u. We shall defer doing so as further manipulations will be required later on that will undo the inverse cosine.

Note that the relation between z and θ (or look-back time) depends on q_0 and has the following sense: for a given redshift z, the difference between the time now and the time of emission $(t_0 - t_i)$ is bigger for a low q_0 universe than it is for a high q_0 universe. Qualitatively, this occurs because in a low q_0 universe, the age of the universe is roughly $1/H_0$, whereas in, say, a $q_0 = 1/2$ universe, the age is only 2/3 as much.

An alternate form of Eq. (6.3) is sometimes useful when dealing with more complex cosmologies, such as those with dark energy, where the solution in time does not admit a simple form involving conformal time. We have $du = dt/R$. From the definition of the Hubble constant $H = (1/R)dR/dt$, we have $dt = dR/(HR)$. From Eq. (6.5), we have $R = R_0/(1+z)$. Combining, we can write $R_0 du = dz/H$. With our expression for $H(z)$, we can integrate this equation to once again compute u as a function of z:

$$R_0 u = \int_0^z \frac{dz'}{H(z')}. \tag{6.14}$$

Angular Diameters and Angular Diameter Distance

Consider the process of observing a distant galaxy with redshift z that subtends an angle ϕ. We ask the question, what is the true linear size of the galaxy. As for the case of special relativity, the process of making a measurement requires that we specify two points in 4-dimensional space-time and compute the proper distance between them. In this case, the only subtle point is that the time coordinate t for the two events (in this case, two points on opposite sides of the galaxy) is t_i, the time of emission, rather than t_0, the time now. The linear separation is given by

$$s = R_i S_k(u)\phi. \tag{6.15}$$

It turns out to be convenient to define an intermediate quantity Z by

$$Z = R_0 H_0 S_k(u) = R_0 H_0 S_k(\theta_0 - \theta_z) = \frac{1}{\sqrt{(2q_0 - 1)k} S_k(\theta_0 - \theta_z)}. \tag{6.16}$$

We have $S_k(\theta_0 - \theta_i) = S_k(\theta_0)C_k(\theta_i) - S_k(\theta_i)C_k(\theta_0)$. Upon substituting Eq. (6.13) and using simple trig identities, we find

$$\begin{aligned} Z =& \frac{1}{q_0^2(1+z)}\left\{q_0 z + (q_0 - 1)[\sqrt{1+2z_0 z} - 1]\right\} \\ =& \frac{z}{1+z}\left[1 + \frac{z(1-q_0)}{\sqrt{1+2q_0 z} + 1 + q_0 z}\right]. \end{aligned} \tag{6.17}$$

Equation (6.15) then becomes (with the aid of Eq. [6.5])

$$s = \frac{R_0 S_k(\theta_0 - \theta_i)\phi}{1+z} = \frac{Z\phi}{H_0(1+z)}. \tag{6.18}$$

We define the angular diameter distance of an object to be

$$D_A = \frac{Z}{H_0(1+z)}. \tag{6.19}$$

The angular diameter of a standard yardstick as a function of redshift is shown in Fig. 6.1 for three different values of q_0. The angular diameter for distant objects is larger than it would be in a Euclidean universe. Two causes are at work. The first is essentially just aberration. The second is the non-Euclidean relationship between radius and circumference in all but a $k = 0$ universe.

Luminosities, fluxes, and surface brightnesses

The luminosity or flux from a source can mean one of several things: it can be the total or bolometric luminosity L_{Bol} (erg sec^{-1}), photon luminosity L_P (photons sec^{-1}), or monochromatic luminosity L_ν (erg sec^{-1} hz^{-1}). We shall consider each in turn.

The easiest to compute is the photon flux. Suppose a source at redshift z and comoving distance u emits δN photons in a time interval δt_i. After the photons have traveled the distance u, they are spread over a surface area $A = 4\pi R_0^2 S_k^2(u)$, where R_0 is the curvature radius at the time t_0 when they arrive at an observer. The time it takes to collect the photons is $\delta t_0 = \delta t_i(1 + z)$. The photon flux is define as

$$F_P = \frac{d^2N}{dtA} = \frac{dN/dt_0}{A} = \frac{dN/dt_i}{4\pi R_0^2 S_k^2(u)(1+z)}, \tag{6.20}$$

with units of photons cm^{-2} s^{-1}. Substituting Z for several of the parameters, we find

$$F_P = \frac{L_P}{4\pi\,(Z/H_0)^2\,(1+z)}. \tag{6.21}$$

The bolometric luminosity is defined as $L_{Bol} = (h\nu_i)dN/dt_i$, where ν_i is some characteristic frequency of the source. At the receving end, the bolometric flux is

$$F_{Bol} = \frac{h\nu_0\,(dN/dt_0)}{A} = \frac{h\nu_i\,(dN/dt_i)}{4\pi R_0^2 S_k^2(u)(1+z)^2} = \frac{L_{Bol}}{4\pi[(Z/H_0)(1+z)]^2}, \tag{6.22}$$

with units of erg sec^{-1} cm^{-2}.

We define the luminosity distance of an object to be

$$D_L = (Z/H_0)(1+z). \tag{6.23}$$

It is $(1+z)^2$ times the angular diameter distance.

The monochromatic luminosity is defined to be $dL/d\nu$ and is a function of frequency; similarly, the monochromatic flux is defined to be $dF_{Bol}/d\nu$. The observed monochromatic flux at frequency ν is given by

$$F_\nu = \frac{L_{\nu(1+z)}(1+z)}{4\pi D_L^2}. \tag{6.24}$$

The appropriate equation to use depends on the type of observation being made. For example, most optical observations measure the flux in a limited bandpass and hence sample the monochromatic luminosity most closely; we will consider such observations more carefully in the next chapter.

Note that

$$\frac{I_\nu}{\nu^3} = \frac{I_{\nu(1+z)}(1+z)^3}{(1+z)^3 \nu_{1+z}^3} = \frac{I_{\nu(1+z)}}{\nu_{1+z}^3} = \frac{I_{\nu'}}{\nu'^3}. \tag{6.25}$$

So I_ν/ν^3 is an invariant. This result is independent of cosmology. As an example, consider observing a blackbody of temperature T that is at a redshift z. The intrinsic intensity of a blackbody is given by

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}. \tag{6.26}$$

The observed intensity is

$$I_\nu = \frac{I_{\nu(1+z)}}{(1+z)^3} = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left[h\nu(1+z)/kT\right] - 1}. \tag{6.27}$$

So we see a blackbody spectrum with temperature $T/(1+z)$. Equivalently, $T_{BB} \propto 1/R$.

The above discussion shows that the concept of "distance" becomes ill-defined when making observations of objects at large redshift. The most sensible definition of distance might be $R_0 u$ where u is the comoving distance of an object corresponding to redshift z. This is the proper distance that would be measured today by the cooperative effort of many F.O.'s placed along the path between the observer and the distant object. However, this distance corresponds to neither the angular diameter distance nor the luminosity distance. Furthermore, the relation between u and z depends on q_0. On the other hand, one can make use of this dependence to devise tests for measuring q_0. This will be one subject in the next chapter.