IX. Dark Energy

In recent years evidence has accumulated that the universe is indeed critically bound as predicted by inflationary cosmology, and while 30% of the current mass density is in the form of baryons or "dark matter" that clusters like galaxies, about 70% is in a form of matter that does not cluster. Furthere there is evidence that the universe is actually accelerating ($q_0 < 0$). The form of this matter is not well understood, but the term "dark energy" has been coined to describe it. In this chapter we revisit the derivations of chapters 5 to 8 and present revised formulations that incorporate the dark energy.

A. Equation of State

Although the nature of the dark energy is completely unknown, the evidence for an accelerating universe forces us to consider a field with an equation of state of the form

$$P = w\rho, \tag{9.1}$$

the same as Eq. (8.1), with $w < 0$. The case $w = -1$ corresponds to constant vacuum energy density. Such an equation of state gives rise to a term in the Einstein field equations that is identical to that of the classic cosmological constant originally introduced by Einstein to produce a static universe. Einstein originally viewed this term as being a modification to the law of gravity and thus rejected it once it was learned that the universe is, in fact, expanding. However, if one interprets the term as arising from a particular form of mass-energy (moving the term from the left side of the equation to the right side) then the motivation for whether to include it or not becomes quite different. Even if the interpretation is different, mass-energy of this form is still referred to as a "cosmological constant."

A constant vacuum energy density leads us to the uncomfortable situation that we are living at a special epoch, because the ratio of ordinary matter density to dark energy density changes with time, and the fact that they are nearly equal today means that we are living at a magic time. This has led theorists to consider theories of dark energy that lead to an equation of state where w is different from -1, and in fact, w need not be constant (although we will not consider such a general case here.) Such mass-energy has been called "quintessence".

B. Dynamics

The energy and acceleration equations has been derived in Eq. (8.6) and (8.7), where we now show the separate contributions of dark matter (including baryons and other zero pressure species) (ρ_M) and dark energy (ρ_Λ) explicitly. For simplicity, we will only consider the case $w = -1$.

$$\ddot{R} = -\frac{4}{3}\pi G(\rho_M - 2\rho_\Lambda), \tag{9.2}$$

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi G(\rho_M + \rho_\Lambda)R^2 - \frac{1}{2}k. \tag{9.3}$$

The dependence of the different types of matter on radius are given by

$$\rho_{DM} = \frac{K}{R^3} \tag{9.4}$$

and

$$\rho_{DE} = \frac{\alpha}{R^{3(1+w)}}. \tag{9.5}$$

Equation (9.3) can be cast in dimensionless form as follows:

1. Let $R_s = c/H_0$;
2. Let the dimensionless radius $x = R/R_s$. Since the volume encompassed by R itself is not yet specified, we do so now by requiring that $x = 1$ today;
3. Let the dimensionless time $\theta = t/H_0$;
4. The critical density today is $\rho_0^{crit} = 3H_0^2/8\pi G$;
5. Let the density of dark matter today $\rho_0^{DM} = \Omega_0^{DM}\rho_0^{crit}$;
6. Let the density of dark energy today $\rho_0^{DE} = \Omega_0^{DE}\rho_0^{crit}$.

With these definitions, we find that $K = \Omega_0^{DM}R_s^3\rho_0^{crit}$ and $\alpha = \Omega_0^{DE}R_s^{3(1+w)}\rho_0^{crit}$. Equation (9.3) reduces to

$$\frac{dx}{d\theta} = \sqrt{\frac{\Omega_0^{DM}}{x} + \frac{\Omega_0^{DE}}{x^{1+3w}}}. \tag{9.6}$$

For arbitrary w, this equation must be integrated numerically. Figure 9.1 shows the dependence of x on θ for a representative set of values of w.

We can once again manipulate Eqs. (8.6) and (8.8) to determine how various quantities such as H and q vary as a function of redshift. The main domain of interest is the present era, when dark energy is becoming dominant, so for simplicity, let us take $w = -1$ (a cosmological constant type universe) for the dark energy component. Equation (8.6) can be written in the form:

$$H^2 = \frac{8}{3}\pi G(\rho_M + \rho_\Lambda) - \frac{k}{R^2}, \tag{9.7}$$

where ρ_M is the density of "cold" matter (baryons plus dark) and ρ_Λ is the density of dark energy. We have also made use of the fact that $\epsilon = -k/2$. Let $\Omega_M = \rho_M/(3H^2/8\pi G)$ and $\Omega_\Lambda = \rho_\Lambda/(3H^2/8\pi G)$. [Note: the classic cosmological constant parameter $\Lambda = 8\pi G\rho_\Lambda$.] Rearranging Eq. (9.7), we find

$$R = \frac{1}{H\sqrt{k(\Omega_M + \Omega_\Lambda - 1)}} \tag{9.8}$$

The dependence of density on redshift is given by

$$\rho_M = \frac{3\Omega_m^0}{8\pi G} H_0^2 (1+z)^3, \tag{9.9}$$

$$\rho_\Lambda = \frac{3\Omega_\Lambda^0}{8\pi G} H_0^2. \tag{9.10}$$

Substituting these into Eq. (9.7), we find

$$H^2 = H_0^2[(1+z)^2(1+\Omega_M^0 z) - \Omega_\Lambda^0 z(2+z)]. \tag{9.11}$$

Finally, from Eqs. (9.2) and (9.11) and the definition of q, we find

$$q = \frac{1}{2}\frac{(1+z)^3\Omega_M^0 - 2\Omega_\Lambda^0}{(1+z)^2(1+\Omega_M^0 z) - \Omega_\Lambda z(2+z)}. \tag{9.12}$$

If $\Omega_\Lambda^0 = 1$ and $\Omega_M^0 = 0$ we have $q_0 = -1$.

C. Observables

Chapter 6 provided the basic recipes for computing various observable quantities; however, many of the equations in that chapter are applicable only to a matter dominated universe. In this section, we indicate the modifications needed to accommodate a dark energy component of the universe. It is important to remember that we are treating dark energy only in the case that the universe is critically bound, whereas Chapter 6 dealt with arbitrary values of the density relative to critical. It would be straightforward to include bound or unbound models in this section, but the increase in parameter space would make the discussion unwieldly.

First, Eq. (6.2), which gives a prescription for computing the comoving distance u to an object, and Eq. (6.5), which relates redshift z to radius R, are still completely valid. Equation (6.15), which defines an intermediate parameter Z, needs additional work, but since this parameter is used explicitly in many of the equations to compute flux, angular diameters, number counts, etc, virtually all the equations in Chapters 6 and 7 that compute physical quantities in terms of Z can be used unchanged.

Equation (6.15) defines Z to be

$$Z = R_0 H_0 S_k(u), \tag{9.13}$$

where u is the comoving distance between an observer today and a distant object at redshift z. This equation omits the replacement of u with conformal time $\theta_0 - \theta_z$ because the latter is valid only for a universe of ordinary matter. A convenient expression for u can be derived by combining Eqs. (6.14) and (9.11). With a bit of rearranging, we find

$$R_0 H_0 u = \int_0^z \frac{dz'}{\sqrt{(1+z')^2(1+\Omega_M^0 z') - \Omega_\Lambda^0 z'(2+z')}}. \tag{9.14}$$

If the universe is not critically bound, we can use Eq. (9.8) to compute $R_0 H_0$:

$$R_0 H_0 = \frac{1}{\sqrt{k(\Omega_M^0 + \Omega_\Lambda^0 - 1)}}. \tag{9.15}$$

If the universe is critically bound, Eq. (9.13) reduces to

$$Z = R_0 H_0 u, \tag{9.16}$$

so Z is given by the right side of Eq. (9.14) directly.

Equation (9.14) must, once again, be integrated numerically. Equations (9.15), (9.13), and (9.14) may be combined with other equations in Chapters 6 and 7 to compute various observable quantities such as bolometric luminosity, number counts, etc. For quantities such as the differential number counts as a function of redshift (Eq. 7.10), there is one extra step necessary:

$$\frac{dN}{dz} = \rho_0 R_0^3 u^2 \frac{du}{dz}. \tag{9.18}$$

We need du/dz. From Eq. (9.14), we find

$$R_0 \frac{du}{dz} = \frac{1}{H_0 \sqrt{(1+z')^2(1+\Omega_M^0 z') - \Omega_\Lambda^0 z'(2+z')}}. \tag{9.19}$$

Plugging in Eqs. (9.14) and (9.19) into (9.18) gives the desired result,

D. Tests of Dark Energy Using Galaxy Clusters

One of the proposed tests for measuring the dark energy parameter w is to count galaxy clusters to high redshift. The equations required are (9.18), which gives the fundamental equation, and (11.xxx), which gives the comoving density of galaxy clusters as a function of their mass. Auxiliary equations include (9.14) and (9.19). Combining everything together, we find that

$$\frac{dN}{d\Omega dz d\ln M} = \frac{\rho_0 c^3}{H_0^3 M_*} u^2 \frac{du}{dz} \sqrt{\frac{2}{\pi}} \frac{1}{D(z)} \left(\frac{M}{M_*}\right)^{\alpha-1} \alpha\, e^{-1/[2D(z)^2]\left(\frac{M}{M_*}\right)^{2\alpha}}. \tag{9.20}$$

Here, N is the cumulative number of clusters as a function of mass, redshift, and angular area on the sky.

The front-end constants can be simplified by making use of the definitions of ρ_0 and M_*. After some manipulation, one finds:

$$\frac{dN}{d\Omega dz d\ln M} = \frac{c}{8H_0} \left(\frac{3}{4\pi}\right) \left(\frac{1.69}{\sigma_8}\right)^{\frac{1}{\alpha}} u^2 \frac{du}{dz} \sqrt{\frac{2}{\pi}} \frac{1}{D(z)} \left(\frac{M}{M_*}\right)^{\alpha-1} \alpha\, e^{-1/[2D(z)^2]\left(\frac{M}{M_*}\right)^{2\alpha}}. \tag{9.21}$$