An Analytical Track Fitter for rapid least-χ^2 fitting

Souvik Das

February 13, 2015

Abstract

An analytical method for fitting points in 3D space to a helical track hypothesis with least χ^2 is described. This method is developed in the context of fitting tracks in the trigger of the CMS experiment where the 3D points correspond to energy deposits left by charged particles in the CMS tracker in the presence of a 3.8 T solenoidal magnetic field. A method to compute the track parameters directly from the 3D coordinates of the energy deposits and their associated uncertainties, that requires minimal computation and may be executed rapidly on an FPGA is described. Part I of the document describes this method for track hypotheses without an impact parameter, separately in the barrel and endcap regions of the tracker. Part II describes this method for the general case of tracks with an impact parameter. In both parts, we illustrate the performance of this fitter with simulated single muon events in the detector. The spirit of this analytical method can be refitted for other tracking detectors in high energy physics where latency is of critical concern.

Contents

The problem of fitting the track of a charged particle through a sequence of experimentally measured points of energy deposits in 3D space is a common one in high energy physics, and several solutions have been devised for it. Kalman Filtering is a common and powerful technique that incorporates the energy loss of the particle as it passes through material media, but it requires significant computation and cannot be employed in hardware-based triggers. A least-χ^2 fit is a simpler approach that sacrifices physical accuracy for the kind of speed required in hardware triggers. In this paper, we describe a least-χ^2 fit algorithm for the proposed track-trigger of the CMS experiment as it prepares for the intense data-taking conditions of the High Luminosity LHC (HL-LHC) with center of mass energies of 14 TeV, peak instantaneous luminosities of 5×10^{34} cm^{-2}s^{-1} and an average of 140 inelastic collisions per bunch crossing. Track-triggering involves a sophisticated flow of information, from the detector readout to its transfer to the counting room where energy deposits (hits) will be detected and tracks reconstructed to be triggered on. The track reconstruction is broken down into two problems – one of finding broad "patterns" of hits that correspond to a track from a bank of all possible patterns using an Associative Memory (AM), and another of computing the track parameters from each selected hit pattern. This paper describes a method for the final computation step.

A detailed description of the CMS apparatus can be found in Ref. [1]. Here, we are concerned with the proposed geometry of the tracker for the HL-LHC period that will have 6 barrel layers and 5 endcap layers of silicon detectors. Each layer will be composed of two sub-layers. The 3 innermost barrel layers will be composed of detector modules with strips of X μm width and X cm length in the outer sub-layer and pixels of X μm width and X cm length in the inner sub-layer. The 3 outermost barrel layers will have both sub-layers composed of the aforementioned silicon strips. Coincidence of two hits that fall within an window of $\delta s =$ X strip widths are used to reconstruct tracks at the trigger, and are called "stubs".

For the case of a single charged track in the detector, the AM may identify it with a certain probability and output a pattern that consists of 5 or 6 stubs. These stubs are characterized in 3-D space by ($r_i^m \pm \Delta r_i^m$, $\phi_i^m \pm \Delta\phi_i^m$, $z_i^m \pm \Delta z_i^m$), where i is the index of the stub in a road, and m is to remind us that these are measured quantities. The uncertainties come from the dimensions of the strips and pixels. r_i^m correspond to the known radii of the layers. The track hypothesis is a helix parametrized by its radius of curvature (ρ), the impact parameter or the distance of closest approach to the z-axis through the central axis of the detector (d_0), the azimuthal angle subtended by the track and x-axis at this point of closest approach (ϕ_0), the z at the point of closest approach (z_0), and the pitch of the helix ($\cot\theta$). The track is analyzed in the $r-\phi$ and $r-z$ planes in order to arrive at parametrized equations for the track connecting ϕ_i to r_i, and z_i to r_i, respectively. Then, the residuals between the hypothesis ϕ_i and measured ϕ_i^m, and between the z_i and z_i^m, are used to define the χ^2 between the track hypothesis and the measured stub coordinates. This χ^2 is minimized analytically in order to arrive at the best-fit track parameters. Part I of this paper assumes $d_0 = 0$, given that the track trigger may never be used to trigger on impact parameters, that are typically of the order of 450 μm, for identifying the decay of b-hadrons. This simplifies the solution considerably. Part II does away with this assumption and reveals the solution for the most general track.

Part I

1 Track Fit without Impact Parameter

1.1 Barrel Region

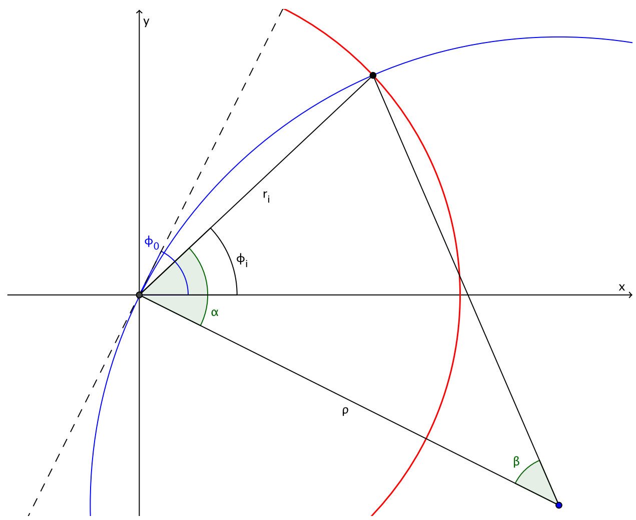

Figure 1: A schematic of a track (blue) in the $r - \phi$ plane, originating from (x =0, y =0) and intersecting with the i^{th} layer of the detector (red). The track, in this plane, is a circle of radius ρ whose tangent at the origin creates an angle of ϕ_0 to the x-axis. The location of the i^{th} stub hypothesis is specified by the radius of the layer it occurs in r_i and its azimuthal angle ϕ_i.

A track with zero impact parameter is a circle in the $r - \phi$ plane that passes through the origin (x =0, y =0), has a radius of ρ, and has a tangent at the origin creates an angle of ϕ_0 to the x-axis. This is illustrated in Fig. 1 where it is shown overlaid with the i^{th} layer of the detector in the barrel region. The intersection of the track with the i^{th} layer of the detector creates the i^{th} stub at (r_i, ϕ_i). The angle subtended by the chord of length r_i from the origin to the stub at the center of the track circle is called β. The angle between the location of the stub, the origin, and the center of the track circle is called α. Equation 1 connects the angle α to the coordinate ϕ_i and the track parameter ϕ_0.

$$\alpha = \frac{\pi}{2} - (\phi_0 - \phi_i) \tag{1}$$

Now, we can connect the r_i and ϕ_i to the track parameters through the simultaneous Eq.2 and 3.

$$r_i \sin\alpha = \rho \sin\beta \tag{2}$$

$$r_i \cos\alpha + \rho\cos\beta = \rho \tag{3}$$

The angle β is eliminated between Eq. 2 and 3 to obtain the relation in Eq. 4.

$$\cos\alpha = \frac{r_i}{2\rho} \tag{4}$$

Eq. 4, along with the relation in 1, allows us to compute ϕ_i as Eq. 5. This serves as the parametric equation for the track in the $r-\phi$ plane.

$$\phi_i = \phi_0 - \arcsin\left(\frac{r_i}{2\rho}\right) \tag{5}$$

To compute z_i of the stub hypothesis as a function of r_i and track parameters, that can serve as the parametric equation of the track in the $r-z$ plane, we eliminate α from Eq. 2 and 3. This results in the relation in Eq. 6. The trigonometric half-angle relation $1-\cos\beta = 2\sin^2(\beta/2)$ is useful in deriving this.

$$\sin(\beta/2) = \frac{r_i}{2\rho} \tag{6}$$

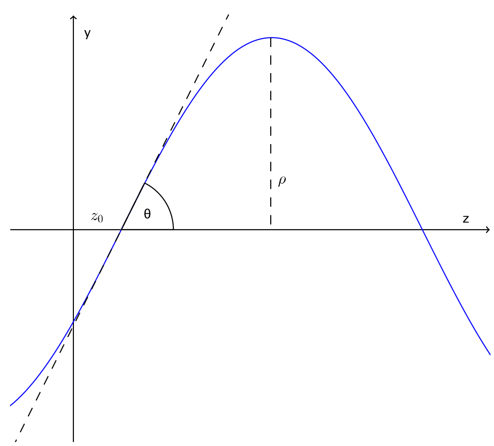

Figure 2: A schematic of a track (blue) originating from (z_0, 0) and intersecting with the i^{th} layer of the detector (red). The track, in this plane, is a sinusoidal curve of amplitude ρ whose tangent at z_0 creates an angle of θ_0 to the z-axis. The location of the i^th stub is specified by the radius r_i of the layer it occurs in, and its z_i coordinate.

The angle β is proportional to the z value on the track. It is related z simply by a wavenumber k as $\beta = k(z-z_0)$, where z_0 is the track's z-intercept. A physical interpretation of k can be sought by projecting the track on the $y-z$ plane as shown in Fig. 2. It is apparent then that k is connected to the slope of the track at z_0 as shown in Eq. 7. θ is the pitch of the track, and $\cot\theta$ is found to serve as a convenient track parameter.

$$\left.\frac{\partial r_i}{\partial z_i}\right|_{z_i=z_0} = \rho k = \tan\theta \tag{7}$$

This allows us to write Eq. 6 in a form that computes z_i as a function of r_i and the track parameters as shown in Eq. 8. The serves as the parametric equation for the track in the $r - z$ plane.

$$z_i = z_0 + 2\rho \cot\theta \arcsin\left(\frac{r_i}{2\rho}\right) \tag{8}$$

Now, armed with parametric equations for the track in $r - \phi$ and $r - z$ planes, we can write the χ^2 as

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{\phi_i^m - \phi_0 + \arcsin(r_i/2\rho)}{\Delta\phi_i^m}\right)^2 + \left(\frac{z_i^m - z_0 - 2\rho\cot\theta \arcsin(r_i/2\rho)}{\Delta z_i^m}\right)^2 \tag{9}$$

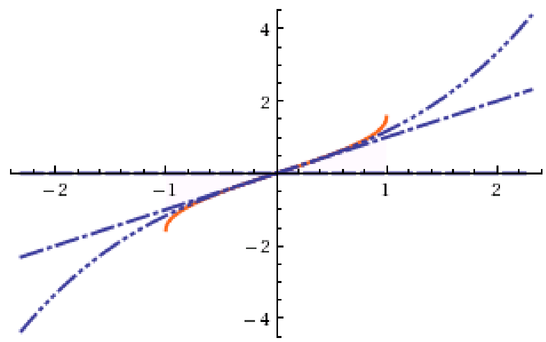

Figure 3: Taylor expansion of the arcsin(x) function around 0.

The ρ of a 2 GeV track in a magnetic field of 3.8 T is approximately 170 cm, while r_6 of the outermost barrel layer is 110 cm. With $r_6/(2\rho) \sim 0.32$, we can refer to the expansion of $\arcsin(x)$ around 0 illustrated in Fig. 3 and conclude that equating $\arcsin(r_i/\rho) = r_i/\rho$ is sufficient in this case. Thus, the equation for χ^2 can be simplified to:

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{\phi_i^m - \phi_0 + (r_i/2\rho)}{\Delta\phi_i^m}\right)^2 + \left(\frac{z_i^m - z_0 - r_i\cot\theta}{\Delta z_i^m}\right)^2 \tag{10}$$

Now we can minimize the expression for χ^2 in Eq. 10 against the track parameters ρ, ϕ_0, $\cot\theta$ and z_0 as shown below.

$$\frac{\partial\chi^2}{\partial\phi_0} = -2\sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + (r_i/2\rho)}{(\Delta\phi_i^m)^2}\right) = 0 \tag{11}$$

$$\frac{\partial\chi^2}{\partial(1/\rho)} = \sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + (r_i/2\rho)}{(\Delta\phi_i^m)^2}\right) r_i = 0 \tag{12}$$

$$\frac{\partial\chi^2}{\partial z_0} = -2\sum_{i=1}^{N}\left(\frac{z_i^m - z_0 - r_i\cot\theta}{(\Delta z_i^m)^2}\right) = 0 \tag{13}$$

$$\frac{\partial\chi^2}{\partial(\cot\theta)} = -2\sum_{i=1}^{N}\left(\frac{z_i^m - z_0 - r_i\cot\theta}{(\Delta z_i^m)^2}\right) r_i = 0 \tag{14}$$

We now solve these equations simultaneously for ρ, ϕ_0, $\cot\theta$ and z_0 by making the following convenient definitions. Here, N is the total number of stubs correponding to a track. Each quantity below may be computed in one for-loop over the stubs corresponding to a single track in a matched pattern.

$$A = \sum_{i=1}^{N} \frac{\phi_i^m}{(\Delta\phi_i^m)^2} \tag{15}$$

$$B = \sum_{i=1}^{N} \frac{1}{(\Delta\phi_i^m)^2} \tag{16}$$

$$C = \sum_{i=1}^{N} \frac{r_i}{(\Delta\phi_i^m)^2} \tag{17}$$

$$E = \sum_{i=1}^{N} \frac{\phi_i^m r_i}{(\Delta\phi_i^m)^2} \tag{18}$$

$$F = \sum_{i=1}^{N} \frac{r_i^2}{(\Delta\phi_i^m)^2} \tag{19}$$

$$G = \sum_{i=1}^{N} \frac{z_i^m r_i}{(\Delta z_i^m)^2} \tag{20}$$

$$H = \sum_{i=1}^{N} \frac{r_i}{(\Delta z_i^m)^2} \tag{21}$$

$$I = \sum_{i=1}^{N} \frac{r_i^2}{(\Delta z_i^m)^2} \tag{22}$$

$$L = \sum_{i=1}^{N} \frac{z_i^m}{(\Delta z_i^m)^2} \tag{23}$$

$$M = \sum_{i=1}^{N} \frac{1}{(\Delta z_i^m)^2} \tag{24}$$

$$P = \sum_{i=1}^{N} \left(\frac{\phi_i^m}{\Delta\phi_i^m} \right)^2 \tag{25}$$

$$Q = \sum_{i=1}^{N} \left(\frac{z_i^m}{\Delta z_i^m} \right)^2 \tag{26}$$

The measurement uncertainties, $\Delta\phi_i^m$ and Δz_i^m, are predetermined to be $1/\sqrt{12}$ of the strip/pixel width and length, respectively. Therefore, the quantities B, C, F, H, I, and M are constants and need only be computed once and not for every track. These definitions allow us to express the simultaneous equations 12-14 simply as follows.

$$A - \phi_0 B + \frac{C}{2\rho} = 0 \tag{27}$$

$$E - \phi_0 C + \frac{F}{2\rho} = 0 \tag{28}$$

$$L - z_0 M - H \cot\theta = 0 \tag{29}$$

$$G - z_0 H - I \cot\theta = 0 \tag{30}$$

And we solve them to obtain formulae for the track parameters as shown below.

$$\phi_0 = \frac{AF - CE}{BF - C^2} \tag{31}$$

$$1/\rho = \frac{2(AC - BE)}{BF - C^2} \tag{32}$$

$$z_0 = \frac{IL - GH}{IM - H^2} \tag{33}$$

$$\cot\theta = \frac{GM - HL}{IM - H^2} \tag{34}$$

The denominators of ϕ_0 and $1/\rho$ are identical, as are that of z_0 and $\cot\theta$. Furthermore, all denominators are constants. This should simplify its implementation in hardware.

The χ^2 of the fit is most usefully separated into χ^2_ϕ and χ^2_z.

$$\chi^2 = \chi^2_\phi + \chi^2_z \tag{35}$$

These two components are calculated as shown below in Eq. 36 and 37.

$$\chi^2_\phi = P + \frac{E}{2\rho} - \phi_0 A \tag{36}$$

$$\chi^2_z = Q - z_0 L - G \cot\theta \tag{37}$$

We can cut on χ^2_z to reduce combinatorial backgrounds before cutting on χ^2_ϕ. The reduced χ^2 for the fit is given as shown in Eq. 38.

$$\frac{\chi^2}{nDOF} = \frac{\chi^2}{N - 4} \tag{38}$$

Part II

2 Track Fit with Impact Parameter

2.1 Barrel Region

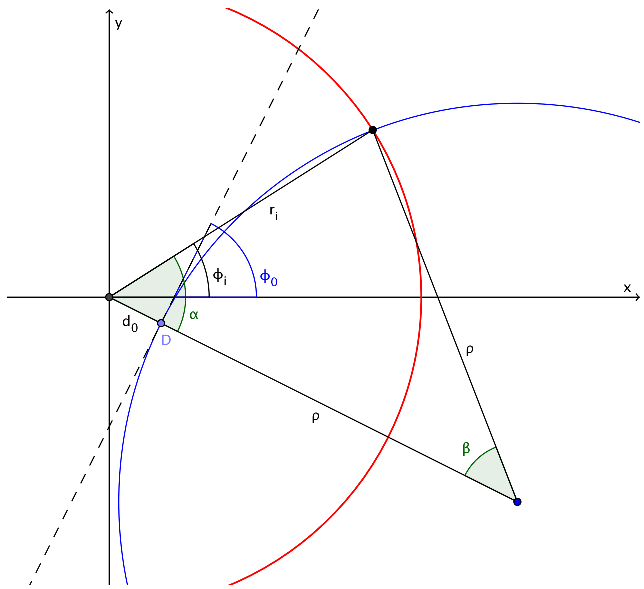

Figure 4: A schematic of a track (blue) in the $r-\phi$ plane whose distance of closest approach to ($x=0$, $y=0$) is d_0, and that intersects with the i^{th} layer of the detector (red). The track, in this plane, is a circle of radius ρ whose tangent at the point of closest approach to the origin creates an angle of ϕ_0 to the x-axis. The location of the i^{th} stub hypothesis is specified by the radius of the layer it occurs in r_i and its azimuthal angle ϕ_i.

A track with impact parameter of d_0 is a circle in the $r-\phi$ plane as shown in Fig. 4. Its center is located at a distance of $d_0+\rho$ from the origin, and its tangent at the distance of closest approach to the origin creates an angle of ϕ_0 to the x-axis. This is illustrated in the figure, where it is shown overlaid with the i^{th} layer of the detector in the barrel region. The intersection of the track with the i^{th} layer of the detector creates the i^{th} stub hypothesis at (r_i, ϕ_i). In the figure, the angle subtended by the chord of length r_i at the center of the track circle is called β. The angle between the location of the stub, the origin, and the center of the track circle is called α. Equation 1 connects the angle α to the coordinate ϕ_i and the track parameter ϕ_0.

We connect the r_i and ϕ_i to the track parameters through the simultaneous equations below.

$$r_i \sin\alpha = \rho \sin\beta \tag{39}$$

$$r_i \cos\alpha + \rho\cos\beta = \rho + d_0 \tag{40}$$

The angle β is eliminated between Eq. 39 and 40 to reveal the relation in Eq. 41.

$$\cos\alpha = \frac{d_0^2 + r_i^2 + 2\rho d_0}{2r_i(\rho + d_0)} \tag{41}$$

This, along with the relation in 1, allows us to compute ϕ_i as Eq. 42.

$$\phi_i = \phi_0 - \arcsin\left(\frac{d_0^2 + r_i^2 + 2\rho d_0}{2r_i(\rho + d_0)}\right) \tag{42}$$

Given that the impact parameter, if any, will be of the order of 400 μm and the radii of the track and the detector layers are of the order of 100 cm, we make the simplifying assumption that $d_0 << \rho, r_i$, and write to first order in d_0/r_i Eq. 43.

$$\phi_i = \phi_0 - \arcsin\left(\frac{r_i}{2\rho} + \frac{d_0}{r_i}\right) \tag{43}$$

Now, in order to compute z_i of the stub hypothesis in the $r - z$ plane, we eliminate α from Eq. 39 and 40 to reveal the relation in Eq: 44.

$$\sin^2(\beta/2) = \frac{r_i^2 - d_0^2}{4\rho(\rho + d_0)} \tag{44}$$

This, along with the relation between β, the pitch θ and z_0 shown in Eq. 7, allows us to compute z_i as shown in Eq 45.

$$z_i = z_0 + 2\rho\cot\theta\arcsin\left(\sqrt{\frac{r_i^2 - d_0^2}{4\rho(\rho + d_0)}}\right) \tag{45}$$

Again, we write this to first order in d_0/ρ as Eq. 46.

$$z_i = z_0 + 2\rho\cot\theta\arcsin\left(\frac{r_i}{2\rho}\left(1 - \frac{d_0}{2\rho}\right)\right) \tag{46}$$

Now, armed with expressions for ϕ_i and z_i in the $r - \phi$ and $r - z$ planes, respectively, in terms of r_i and the track parameters, we can write the χ^2 as

$$\chi^2 = \sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + \arcsin\left(\frac{r_i}{2\rho} + \frac{d_0}{r_i}\right)}{\Delta\phi_i^m}\right)^2 + \left(\frac{z_i^m - z_0 - 2\rho\cot\theta\arcsin\left(\frac{r_i}{2\rho}\left(1 - \frac{d_0}{2\rho}\right)\right)}{\Delta z_i^m}\right)^2 \tag{47}$$

where ϕ_i^m and z_i^m are the measured coordinates of the i^{th} stub and N is the total number of stubs in a pattern. As motivated in Section 1, we can replace $\arcsin(x)$ with x for tracks with momentum greater than 3 GeV. Thus, an approximate version of the χ^2 becomes

$$\chi^2 = \sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + \frac{r_i}{2\rho} + \frac{d_0}{r_i}}{\Delta\phi_i^m}\right)^2 + \left(\frac{z_i^m - z_0 - r_i\cot\theta\left(1 - \frac{d_0}{2\rho}\right)}{\Delta z_i^m}\right)^2 \tag{48}$$

Now we can minimize the expression for χ^2 in Eq. 48 with respect to the track parameters ϕ_0, $1/\rho$, d_0, z_0, and $\cot\theta$ as shown below.

$$\frac{\partial\chi^2}{\partial\phi_0} = -2\sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + \frac{r_i}{2\rho} + \frac{d_0}{r_i}}{(\Delta\phi_i^m)^2}\right) = 0 \tag{49}$$

$$\frac{\partial\chi^2}{\partial(1/\rho)} = \sum_{i=1}^{N}\left(\frac{\phi_i^m - \phi_0 + \frac{r_i}{2\rho} + \frac{d_0}{r_i}}{(\Delta\phi_i^m)^2}\right) r_i - \left(\frac{z_i^m - z_0 - r_i\cot\theta\left(1 - \frac{d_0}{2\rho}\right)}{(\Delta z_i^m)^2}\right) r_i d_0 \cot\theta = 0 \tag{50}$$

$$\frac{\partial\chi^2}{\partial d_0} = \sum_{i=1}^{N} 2\left(\frac{\phi_i^m - \phi_0 + \frac{r_i}{2\rho} + \frac{d_0}{r_i}}{(\Delta\phi_i^m)^2}\right)\frac{1}{r_i} + \left(\frac{z_i^m - z_0 - r_i\cot\theta\left(1 - \frac{d_0}{2\rho}\right)}{(\Delta z_i^m)^2}\right)\left(\frac{r_i\cot\theta}{\rho}\right) \tag{51}$$

$$\frac{\partial\chi^2}{\partial z_0} = -2\sum_{i=1}^{N}\left(\frac{z_i^m - z_0 - r_i\cot\theta\left(1 - \frac{d_0}{2\rho}\right)}{(\Delta z_i^m)^2}\right) = 0 \tag{52}$$

$$\frac{\partial\chi^2}{\partial(\cot\theta)} = -2\left(1 - \frac{d_0}{2\rho}\right)\sum_{i=1}^{N}\left(\frac{z_i^m - z_0 - r_i\cot\theta\left(1 - \frac{d_0}{2\rho}\right)}{(\Delta z_i^m)^2}\right) r_i = 0 \tag{53}$$

We solve this set of non-linear equations for the track parameters by making the following convenient definitions. Each quantity can be computed in a for-loop over the coordinates corresponding to the set of stubs for a track.

$$A = \sum_{i=1}^{N} \frac{\phi_i^m}{(\Delta\phi_i^m)^2} \tag{54}$$

$$B = \sum_{i=1}^{N} \frac{1}{(\Delta\phi_i^m)^2} \tag{55}$$

$$C = \sum_{i=1}^{N} \frac{r_i}{(\Delta\phi_i^m)^2} \tag{56}$$

$$D = \sum_{i=1}^{N} \frac{1}{r_i(\Delta\phi_i^m)^2} \tag{57}$$

$$E = \sum_{i=1}^{N} \frac{\phi_i^m r_i}{(\Delta\phi_i^m)^2} \tag{58}$$

$$F = \sum_{i=1}^{N} \frac{r_i^2}{(\Delta\phi_i^m)^2} \tag{59}$$

$$G = \sum_{i=1}^{N} \frac{z_i^m r_i}{(\Delta z_i^m)^2} \tag{60}$$

$$H = \sum_{i=1}^{N} \frac{r_i}{(\Delta z_i^m)^2} \tag{61}$$

$$I = \sum_{i=1}^{N} \frac{r_i^2}{(\Delta z_i^m)^2} \tag{62}$$

$$J = \sum_{i=1}^{N} \frac{\phi_i^m}{r_i(\Delta\phi_i^m)^2} \tag{63}$$

$$K = \sum_{i=1}^{N} \frac{1}{r_i^2(\Delta\phi_i^m)^2} \tag{64}$$

$$L = \sum_{i=1}^{N} \frac{z_i^m}{(\Delta z_i^m)^2} \tag{65}$$

$$M = \sum_{i=1}^{N} \frac{1}{(\Delta z_i^m)^2} \tag{66}$$

$$P = \sum_{i=1}^{N} \left(\frac{\phi_i^m}{\Delta\phi_i^m}\right)^2 \tag{67}$$

$$Q = \sum_{i=1}^{N} \left(\frac{z_i^m}{\Delta z_i^m}\right)^2 \tag{68}$$

The measurement uncertainties, $\Delta\phi_i^m$ and Δz_i^m, are predetermined to be $1/\sqrt{12}$ of the strip/pixel width and length, respectively. Therefore, the quantities B, C, D, F, H, I, K, and M are constants and need only be computed once and not for every track. These definitions allow us to express the simultaneous equations 49-53 simply as follows.

$$A - \phi_0 B + \frac{C}{2\rho} + d_0 D = 0 \tag{69}$$

$$E - \phi_0 C + \frac{F}{2\rho} + d_0 B + (d_0 \cot\theta)G - (z_0 d_0 \cot\theta)H - d_0 \cot^2\theta \left(1 - \frac{d_0}{2\rho}\right) I = 0 \tag{70}$$

$$J - \phi_0 D + \frac{B}{2\rho} + d_0 K + \left(\frac{\cot\theta}{2\rho}\right) G - \left(\frac{z_0 \cot\theta}{2\rho}\right) H - \frac{\cot^2\theta}{2\rho} \left(1 - \frac{d_0}{2\rho}\right) I = 0 \tag{71}$$

$$L - z_0 M - \cot\theta \left(1 - \frac{d_0}{2\rho}\right) H = 0 \tag{72}$$

$$G - z_0 H - \cot\theta \left(1 - \frac{d_0}{2\rho}\right) I = 0 \tag{73}$$

And we solve them to obtain formulae for the track parameters as shown below.

$$\phi_0 = \frac{AB^2 - BDE - BCJ + DFJ + CEK - AFK}{B^3 - 2BCD + D^2F + C^2K - BFK} \tag{74}$$

$$1/\rho = \frac{2(ABD - D^2E - B^2J + CDJ - ACK + BEK)}{B^3 - 2BCD + D^2F + C^2K - BFK} \tag{75}$$

$$d_0 = \frac{ABC - B^2E + CDE - ADF - C^2J + BFJ}{B^3 - 2BCD + D^2F + C^2K - BFK} \tag{76}$$

$$z_0 = \frac{IL - GH}{IM - H^2} \tag{77}$$

$$\cot\theta = \frac{GM - HL}{IM - H^2} / \left(1 - \frac{d_0}{2\rho}\right) \tag{78}$$

A single expression for the denominator appears for ϕ_0, $1/\rho$ and d_0. Furthermore, it is a constant. The denominator for z_0 is also a constant. For $\cot\theta$, $MI - H^2$ is also a constant, and we may choose not to divide by $(1 - d_0/2\rho)$ for greater speed and reduced accuracy. These features of the analytical solution are expected to dramatically simplify its implementation in hardware.

The χ^2 of the fit can also be computed from the aforementioned definitions, as well as the $\chi^2/nDOF$, as shown below in Eq. 79 and 80.

$$\begin{aligned} \chi^2 = P + \left(\phi_0^2 + \frac{d_0}{\rho}\right) B + \frac{F}{4\rho^2} + d_0^2 K - 2\phi_0 A + \frac{E}{\rho} + 2d_0 J - \frac{C\phi_0}{\rho} - 2d_0\phi_0 D \\ + Q + z_0^2 M + I \left(\cot\theta \left(1 - \frac{d_0}{2\rho}\right)\right)^2 - 2z_0 L - 2G\cot\theta \left(1 - \frac{d_0}{2\rho}\right) + 2z_0 H \cot\theta \left(1 - \frac{d_0}{2\rho}\right) \end{aligned} \tag{79}$$

$$\frac{\chi^2}{nDOF} = \frac{\chi^2}{N - 5} \tag{80}$$

References

[1] S. Chatrchyan et al. The CMS experiment at the CERN LHC. *JINST*, 3:S08004, 2008.