Lecture 2 supplement: Representations of Accelerator Elements
maps, kicks and beam propagation

0.1 Symplectic Condition

Let M denote a $2m \times 2m$ matrix. Let J denote the $2m \times 2m$ matrix given by.

$$J = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_1 \end{pmatrix}, \tag{1}$$

where J_1 is the 2×2 matrix given by

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

The matrix M is said to be a *symplectic matrix* if it satisfies

$$\tilde{M} J M = J. \tag{3}$$

Symplectic matrices have several useful properties:

1. $\det M = 1$.
2. The eigenvalues of M are real or they occur in complex conjugate pairs.
3. If λ is an eigenvalue of M, then so is $1/\lambda$.
4. The real dimensionality of M (i.e. the number of real parameters necessary to specify an arbitrary $2m \times 2m$ symplectic matrix) is $m(2m+1)$.
5. The set of $2m \times 2m$ symplectic matrices forms a group, Sp(2m).

0.1.1 Normal Forms

Another important property is related to the *normal form* of a symplectic matrix. A common concept from linear algebra is that, given a real symmetric matrix, it is possible to perform a similarity transformation that transforms it to a diagonal matrix with eigenvalues on the diagonal. Analogously, it is possible to perform a similarity transformation on a symplectic matrix that transforms it into a special form. To connect this to beam dynamics, consider an m-dimensional ($2m$ degree-of-freedom) periodic system, and suppose that the linear dynamics of the system is described by a symplectic matrix M. Suppose that the motion is stable in all m dimensions. Then it is possible to perform a symplectic similarity transformation which turns M into a matrix consisting of 2×2 rotation matrices along the diagonal, and zeros elsewhere. In other words, there exists a symplectic matrix A such that

$$A^{-1} M A = N, \tag{4}$$

where

$$N = \begin{pmatrix} R(\mu_1) & 0 & 0 & 0 \\ 0 & R(\mu_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & R(\mu_n) \end{pmatrix}, \tag{5}$$

and where

$$R(\mu_i) = \begin{pmatrix} \cos\mu_i & \sin\mu_i \\ -\sin\mu_i & \cos\mu_i \end{pmatrix}. \tag{6}$$

Here the eigenvalues of M are $e^{\pm i\mu_i}$. For a periodic system, the eigenvalues are related to quantities of physical interest. Namely, μ_i equals the phase shift per period or the phase shift per cell in the ith dimension.

0.1.2 Mappings

So far we have discussed symplectic matrices. Now we will discuss symplectic mappings. Let ζ denote a vector with $2m$ components, and let $\mathcal{M}$ denote a mapping that maps an initial set of variables, ζ^{in} into a final set of variables ζ^{fin}:

$$\mathcal{M} : \zeta^{in} \to \zeta^{fin}. \tag{7}$$

Let M denote the Jacobian matrix of $\mathcal{M}$:

$$M_{ab} = \frac{\partial \zeta_a^{fin}}{\partial \zeta_b^{in}} \qquad (a, b = 1, \ldots, 2m). \tag{8}$$

Then $\mathcal{M}$ is said to be a *symplectic mapping* if its Jacobian is a symplectic matrix.

0.2 Expansion of Transfer Maps

Consider an m-dimensional dynamical system governed by some Hamiltonian $H(\zeta, t)$, where $\zeta = (q_1, p_2, q_2, p_2, \ldots, q_m, p_m)$. An obvious method of representing $\mathcal{M}$ (or the action of $\mathcal{M}$ on ζ^{in}) is to expand ζ^{fin} as a power series in ζ^{in}:

$$\zeta_a^{fin} = \sum_{b=1}^{2m} M_{ab}\zeta_b^{in} + \sum_{1\le b\le c}^{2m} T_{abc}\zeta_b^{in}\zeta_c^{in} + \sum_{1\le b\le c\le d}^{2m} u_{abcd}\zeta_b^{in}\zeta_c^{in}\zeta_d^{in} + \cdots \tag{9}$$

(Here we have assumed that $\mathcal{M}$ maps the origin of phase space into itself, so there is no constant term in the expansion.) The linear behavior of $\mathcal{M}$ is represented by the matrix M, and the nonlinear behavior is governed by the tensors T, U, etc. For example, the tensor T governs second order nonlinear behavior, the tensor U governs third order behavior, and so on.

If the Taylor series is truncated at some order, then the truncated series is generally not symplectic.

Do not confuse the above expansion with the expansion of the Hamiltonian around the reference trajectory:

$$H = H_2 + H_3 + H_4 + \cdots \tag{10}$$

As already mentioned (in lecture_02 proper), H_2 results to a linear transfer map. This is a "good" representation of our accelerator for *small amplitudes*, i.e no big excursions from the reference trajectory. As we saw in the end of lecture_02 proper, we can apply formula 9 to any truncation of equation 10.

A concrete example: Let's say we would like to compute a third order Taylor map of a lattice section which contains a sextupole. What we have to do is expand around the reference trajectory up to fourth order (equation 10) and then compute the Taylor map from equation 9 to third order in ζ