Notes on the Production of Flat Beams

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1 Introduction

In a recent report, Brinkmann, Derbenev, and Flöttman[1] have proposed a method of delivering a flat electron beam from the source, thus potentially obviating the necessity of a damping ring for electrons in a linear electron-positron collider. In the Autumn of 1999, Helen Edwards and collaborators observed a hint of this effect at Fermilab[2].

In this note, I wish first to develop the derivation presented by the authors cited above in language more familiar to me. I am not doing anything original here; I consider it an important result which deserves translation into other language to aid in its understanding. Then, even though the currently installed equipment at the A0 laboratory in Fermilab is not ideal for the purpose, I want to comment on experiments that can be done to confirm their predictions. Sergei Nagaitsev[3] has already proposed one configuration.

2 Basic Method

2.1 Sequence of Steps

1. A zero-emittance beam from a cathode immersed in a solenoidal field developes an angular momentum at exit from the solenoid.

2. Pass the beam after exit through a quadrupole channel with a 90^0 phase difference between the two transverse degrees of freedom, with scale length defined by the solenoid field. For a skew quadrupole channel, the beam can be flat in either transverse coordinate.

2.2 Simple Version

2.2.1 The Solenoid

Suppose that a cathode is inside a solenoid that produces a field B_z, where z is the beam axis. Further, assume that the beam produced from the cathode has zero emittance. Then the particles just travel along the field lines as they undergo acceleration within the gun, and nothing interesting happens until the end of the solenoid is encountered.

A particle having charge e and momentum p_0 exiting with transverse displacements $x = x_0$ and $y = 0$ will be deflected through an angle in the y direction:

$$\Delta y' = \frac{1}{(p_0/e)} \int B_x dz. \tag{1}$$

Since $\nabla \cdot B = 0$, application of Gauss' Law to a cylindrical surface of radius x_0 that stretches through the end field gives

$$2\pi x_0 \int B_x dz = \pi x_0^2 B_z \tag{2}$$

and so

$$\Delta y' = k x_0; \quad k \equiv \frac{1}{2} \frac{B_z}{(p_0/e)}. \tag{3}$$

Similarly, a particle displaced at $y = y_0$ and $x = 0$ will experience a deflection

$$\Delta x' = -k y_0. \tag{4}$$

In Eqs. 1 and 4, the approximation has been made that the change in the transverse coordinates through the end of the solenoid can be neglected.

Looking downstream, the beam has taken on a clockwise rotation; an angular momentum or vorticity, if you like. The initial state of a particle as it exits the solenoid becomes

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_0 = \begin{pmatrix} x_0 \\ -k y_0 \\ y_0 \\ k x_0 \end{pmatrix}. \tag{5}$$

2.2.2 The Quadrupole Channel

Next pass the beam through an alternating gradient quadrupole channel. Assume that the channel is represented by an identity matrix in the x-direction and has an additional 90^0 phase advance in y. A simple form for the 90^0 matrix is

$$\begin{pmatrix} 0 & \beta \\ -\frac{1}{\beta} & 0 \end{pmatrix} \tag{6}$$

where β is the amplitude function intrinsic to the channel, treated as a repetitive structure.

We get the output state

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\frac{1}{\beta} & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ -k y_0 \\ y_0 \\ k x_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -k y_0 \\ k \beta x_0 \\ -\frac{1}{\beta} y_0 \end{pmatrix}. \tag{7}$$

If now we choose $\beta = 1/k$ the particles end up with equal displacements in x and y and travelling at equal angles in x and y. This describes a flat beam inclined at an angle of 45^0 to the coordinate axes. Change to a skew-quadrupole channel, and the flat beam can be aligned along either the horizontal or vertical axis in the way that these coordinates are usually oriented in the laboratory.

If, between the solenoid end and entry to the quadrupole channel, the beam undergoes a longitudinal acceleration from p_0 to p_1 without transverse momentum change, then the initial state as given in Eq. 5 differs only in that the parameter k is to be evaluated at momentum p_1.

2.3 A More General Treatment

Let's repeat the preceding calculation in somewhat greater generality and also arrive at a flat beam in the more usual laboratory horizontal and vertical coordinates. The 4×4 transport matrix from the end of the solenoid through the skew quadrupole channel can be written in the form

$$M = R^{-1} T R, \tag{8}$$

where R is a coordinate rotation of 45^0 about the longitudinal axis:

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \tag{9}$$

and I is the 2×2 identity matrix. In the rotated coordinates, T represents a normal quadrupole channel, and so can be written

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{10}$$

where A and B are 2×2 matrices. Using Eqs. 9 and 10. Eq. 8 becomes

$$M = \frac{1}{2} \begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix}. \tag{11}$$

Rewrite the initial state given by Eq. 5 in terms of two-element column vectors:

$$X \equiv \begin{pmatrix} x_0 \\ -ky_0 \end{pmatrix} \quad Y \equiv \begin{pmatrix} y_0 \\ kx_0 \end{pmatrix}. \tag{12}$$

The relation between X and Y can be expressed in the form

$$Y = SX; \quad S \equiv \begin{pmatrix} 0 & -\frac{1}{k} \\ k & 0 \end{pmatrix}. \tag{13}$$

With use of Eq. 13, the final state is

$$\begin{pmatrix} X \\ Y \end{pmatrix}_1 = \frac{1}{2} \begin{pmatrix} [A+B+(A-B)S]\,X \\ [A-B+(A+B)S]\,X \end{pmatrix}, \tag{14}$$

The condition for a flat beam in x is that Y_1 vanish. Since x_0 and y_0 are independent variables, this condition implies

$$A - B + (A + B)S = 0, \tag{15}$$

or

$$I = -(A - B)^{-1}(A + B)S \tag{16}$$

Using the Courant-Snyder parameterization[4], A and B can be represented in the form $\exp(J\mu)$ where J is the 2×2 matrix

$$J = \begin{pmatrix} \alpha & \beta \\ -\frac{1+\alpha^2}{\beta} & -\alpha \end{pmatrix}, \tag{17}$$

and μ is the phase advance. Assume that the J matrices of A and B are identical, and that the phase advance of B exceeds that of A by Δ. Then Eq. 16 becomes

$$I = -\frac{cos(\Delta/2)}{sin(\Delta/2)} JS = -\frac{cos(\Delta/2}{sin(\Delta/2)} \begin{pmatrix} k\beta & \frac{-\alpha}{k} \\ -k\alpha & \frac{1+\alpha^2}{k\beta} \end{pmatrix} \tag{18}$$

from which it follows that $\alpha = 0$, $\beta = 1/k$, and $\Delta = 90^0$.

If acceleration occurs between the solenoid end and the quadrupole channel, β_0 is evaluated at the momentum of entry to the channel, as discussed at the end of Sec. 2.2.2.

3 Implementation with a Quadrupole Triplet

3.1 Symmetric Triplet

Brinkmann *et al*[1] use a symmetric quadrupole triplet as an example. I would like to go through their example in some detail because it is a good illustration of how this process works. Let's use thin lenses, and quadrupole strengths will be denoted by the symbol q, where $q > 0$ is the reciprocal of the focal length.

The initial value of the amplitude function, β_0, is given by the solenoid parameter k according to $\beta_0 = 1/k$, and as shown above, the initial value of its slope is characterized by $\alpha_0 = 0$. The first lens of the triplet will be some distance $D1$ from the end of the solenoid. At the entry to this lens, we have

$$\beta_1 = \beta_0 + D1^2/\beta_0 \tag{19}$$

$$\alpha_1 = -D1/\beta_0 \tag{20}$$

and at the exit

$$\alpha_{1x} = \alpha_1 - q_1\beta_1 \tag{21}$$

$$\alpha_{1y} = \alpha_1 + q_1\beta_1 \tag{22}$$

$$
\begin{aligned}
\gamma_{1x} &= \frac{1+\alpha_{1x}^2}{\beta_1} && (23)\\
\gamma_{1y} &= \frac{1+\alpha_{1y}^2}{\beta_1} && (24)
\end{aligned}
$$

where, although we are in rotated coordinates, we retain the subscipts x and y. Below, we will require that the phase advance in y will exceed that in x through the triplet by 90^0.

Therefore after the first lens

$$
\begin{aligned}
\beta_x &= \beta_1 - 2\alpha_{1x}z + \gamma_{1x}z^2 && (25)\\
\beta_y &= \beta_1 - 2\alpha_{1y}z + \gamma_{1y}z^2 && (26)
\end{aligned}
$$

All the action takes place between the first and the second lens. To establish a 90^0 phase difference through the symmetric triplet, a 45^0 phase difference must be developed in the distance, $D2$. between the two lenses. This condition is

$$
\begin{aligned}
\int_0^{D2}\left(\frac{1}{\beta_y(z)} - \frac{1}{\beta_x(z)}\right)dz &= && (27)\\
\arctan\left(\frac{D2}{\beta_1 - \alpha_{1y}D2}\right) - \arctan\left(\frac{D2}{\beta_1 - \alpha_{1x}D2}\right) &= \frac{\pi}{4}. && (28)
\end{aligned}
$$

Think of the second lens as a pair of colocated half lense, each of strength q_2. After the first element of this pair, we want $\alpha = 0$ in both of the transverse degrees of freedom so that the exit values from the triplet will be the same as the entry values. The condition that this be so is

$$
\frac{\alpha_x(D2)}{\alpha_y(D2)} = -\frac{\beta_x(D2)}{\beta_y(D2)}. \qquad (29)
$$

Eqs. 28 and 29 require solution for the two "unknowns" q_1 and $D2$. I have not been able to find a closed-form solution, but have had to use numerical methods. However they are solved, the one remaining parameter of the triplet, q_2, follows from either

$$
\begin{aligned}
\alpha_x(D2) + q_2\beta_x(D2) &= 0, && (30)\\
\alpha_y(D2) - q_2\beta_y(D2) &= 0 && (31)
\end{aligned}
$$

which is where Eq. 29 came from.

3.2 Arrangement at FNPL

We don't have a symmetric triplet, so the preceding analysis does not tell us how to adjust the quadrupole strengths. I was not able to extend the Courant-Snyder parameter based approach to this case in any useful manner.

Until the first quadrupole is encountered, the system has rotational symmetry about the beam axis. The relation between X and Y stated in Eq. 13,

$Y = SX$ remains valid, but S will not longer have the simple form given there. It will also include RF focusing effects.

Nevertheless, all we need for the moment is that a matrix S exists at any point after the booster cavity, up to the first skew quadrupole. So defined, S may include normal quadrupoles. Following the notation of the symmetric case, let the three quadrupole strengths, $B'\ell/(B\rho)$, be q_1, q_2, q_3 separated by distances $D2$ and $D3$ in downstream progression.

The matrix A of Eq. 15 is then

$$A = \begin{pmatrix} 1 & 0 \\ q_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & D3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & D2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}, \tag{32}$$

with B differing in that all the q_i change sign. From Eq. 32,

$$\frac{A+B}{2} = \begin{pmatrix} 1 + D2D3q_1q_2 & D2 + D3 \\ D2q_1q_2 + D2q_1q_3 + D3q_1q_3 + D3q_2q_3 & 1 + D2D3q_2q_3 \end{pmatrix} \tag{33}$$

$$\frac{A-B}{2} = \begin{pmatrix} D2q_1 + D3q_1 + D3q_2 & D2D3q_2 \\ q_1 + q_2 + q_3 + D2D3q_1q_2q_3 & D2q_2 + D2q_3 + D3q_3 \end{pmatrix} \tag{34}$$

The $1,1$ and $1,2$ elements of Eq. 15 may be solved for q_1 and q_2. For q_1, I get[1]

$$q_1^2 + \frac{1 - |S|}{s_{1,2}} q_1 + \frac{D2s_{1,1} - s_{1,2} + D2(D2 + D3)s_{2,1} - (D2 + D3)s_{2,2}}{D2(D2 + D3)s_{1,2}} = 0, \tag{35}$$

where the $s_{i,j}$ are the elements of the matrix S and $|S|$ stands for its determinant.

The second term in Eq. 35 vanishes because $|S| = 1$. At the exit from the solenoid, we had $Y_0 = S_0X_0$, where I've added some subscripts to Eq. 13. Now we pass X and Y through non-coupling transformations M_x and M_y respectively. These transformations include the booster cavity. At entrance to the first skew quadrupole, $Y = M_yS_0M_x^{-1}$; i.e., $S = M_yS_0M_x^{-1}$. Since the determinant of a product of matrices is the product of their individual determinants, the product of Lorentz factors introduced by the forward and backward transformations through the booster cavity results in unit determinant.

Therefore, the solution to Eq. 35is

$$q_1 = \pm\sqrt{-\frac{D2s_{1,1} - s_{1,2} + D2(D2 + D3)s_{2,1} - (D2 + D3)s_{2,2}}{D2(D2 + D3)s_{1,2}}}, \tag{36}$$

q_2 follows from

$$q_2 = -\frac{s_{1,2} + (D2 + D3)s_{2,2}}{D2D3(1 + q_1s_{1,2})}, \tag{37}$$

and q_3 can be found from either term in the second row of Eq. 15. The sign selection in Eq. 36 depends the plane in which you wish the beam to be flat.

This foregoing paragraphs of this subsection were written in the context of predicting settings for the skew quadrupoles given knowledge of the correlations

[1]With thanks to Yin-e Sun for algebraic repairs.

in the incident beam expressed by the matrix S. At present the situation is that we know the skew quadrupole settings for production of a flat beam rather well from our measurements, whereas there are some remaining ambiguities in the experimental determination of the transfer matrix through the booster cavity. It could be useful to reverse the process and see what we can learn about S by inverting Eq. 15.

4 Data Analysis[5]

4.1 General Approach

The rms emittance in one transverse degree of freedom is defined by

$$\varepsilon \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}. \tag{38}$$

At FNPL, the slit method of emittance measurement is used. At X3, a horizontal slit system may be inserted for round beam measurements. At XL6 both horizontal and vertical slits may be inserted. The slits have an aperture of 50μm, 1 mm spacing, and thickness of 6 mm in the beam direction.

If the beam were so large that slit aperture and spacing were negligible, many downstream slit images could be used to evaluate Eq. 38, including the $\langle xx' \rangle$ correlation term. This is not the case for the narrow dimension of the flat beam. Because the OTR screens are viewed from the side, the beam downstream of the skew quadrupole system is oriented so that the narrow dimension, x, is horizontal. On XL6, the beam is only a few hundred μm wide so (typically) only a single slit is illuminated.

In the broad (vertical) dimension, a number of the horizontal slits are illuminated. If the angular momentum has been removed, the images will be horizontal and information about the correlation term in this degree of freedom can be obtained from the lateral offset of the images. In the data that I've seen, this offset is negligible. So for the purposes of this experiment, I'll assume that the correlation term in Eq. 38 can be neglected and the emittance in each degree of freedom is given by

$$\varepsilon = \sqrt{\langle x^2 \rangle} \sqrt{\langle x'^2 \rangle_1}, \tag{39}$$

where the subscript on the angular term indicates that the angle information is obtained from a single slit image.

For this single image, we have

$$\langle x^2 \rangle_1 = \langle x_0^2 \rangle + 2\langle x_0 x' \rangle + L^2 \langle x^2 \rangle_1. \tag{40}$$

Here, L is the distance from the slit to the downstream viewing screen, the subscript on x means that it is measured at the slit, and no subscript is placed on x' since I assume that there is no transverse deflection between slit and screen.

If we assume that the slit of width w is uniformly illuminated, then $\langle x_0^2 \rangle = w^2/12$, and Eq. 40 turns into

$$\langle x'^2 \rangle_1 = \frac{1}{L^2} \left[\langle x^2 \rangle_1 - \frac{w^2}{12} - 2\langle x_0 x' \rangle_1 \right]. \tag{41}$$

In the error analysis we'll need a way of putting a limit on the single slit correlation term.

To simplify the expressions below, the rms deviations from the mean at the position of the slits will be denoted by s_x, s_y, with x the narrow dimension; s'_x, s'_y will be the rms deviations from the mean in angle from Eq. 41. The flat beam emittance ratio is given by

$$R_{yx} = \frac{s_y s'_y}{s_x s'_x} = \frac{\varepsilon_y}{\varepsilon_x}. \tag{42}$$

In R_{yx}, some cancellation of resolution terms takes place, in comparison with the emittances ss' themselves.

Interesting though R_{yx} is by itself (demonstration of a significant value for it was the goal of the first phase of the experiment), the other important ratio is a measure of how well the round beam emittance is preserved. This means that it is necessary to measure the round beam emittance, ε_r, under the same conditions as those during the determination of R_{yx}, and calculate the ratio

$$R_{fr} = \frac{\sqrt{\varepsilon_x \varepsilon_y}}{\varepsilon_r}. \tag{43}$$

In stating emittances, the more familiar normalized variant would be used, obtained by multiplying any ε above by $\gamma(v/c)$.

4.2 Remarks on Procedure

A beam pulse contains a number of bunches, thus each image represents a superposition of these bunches. The charge varies from bunch to bunch, so the charge associated with an image is a range rather than a value. More on this later under the heading of errors.

The file of images for a screen contains a number of order 10 with the laser shutter open and a similar number with laser shutter closed (dark current). Each image contains a random background. In order to suppress the background, a few pictures with beam are averaged as are a few with dark current. The beam images may be adjusted in position or orientation to achieve superposition.

The dark current average is then subtracted from the beam average. Given the 0–255 grey scale, some background locations will inevitably go "negative" and appear as bright spots. Since the brightest spot in the beam tends to be much less than 255, some number (e.g., 50) can be added to each pixel to eliminate the bright spots.

Next, projections are taken in the two degrees of freedom. The entire image is used, unless there is some feature such as strong s-shape that can be reasonably excluded. The distributions associated with the projections may require baseline adjusted. The rms deviations from the mean may then be calculated.

References

[1] R. Brinkmann, Ya. Derbenev, K. Flöttmann, *A Flat Beam Electron Source for Linear Colliders.* TESLA 99-09, April 1999

[2] Helen T. Edwards, private communication

[3] Sergei Nagaitsev, *Vortex-to-plane beam adapter*, private communication, December 2, 1999.

[4] E. D. Courant and H. S. Snyder, *Theory of the Alternating Gradient Synchroton*, Ann. Phys. *3*, 1 (1958).

[5] This section is based on discussions with Helen Edwards, Court Bohn, and Kai Desler.