Some Notes on Longitudinal Emittance

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This note is in response to some recent confusion about definitions of longitudinal emittance, and its relationship to momentum spread, bunch length, etc. In what follows, the derivations of relationships between the rms momentum spread of a presumed Gaussian bunch distribution and the 95% longitudinal emittance are provided. The relationships are valid for a distribution where the emittance is much smaller than the RF bucket area, and therefore effects due to the nonlinearities of the RF focusing and large tails of the particle distribution are ignored.

We begin with the first integral of the equation of motion for particles circulating the accelerator, namely Eq. (2.45) in Edwards and Syphers [1]:

$$\Delta E^2 + \frac{2v^2 E_s eV}{\eta \omega_{rf} \tau c^2}(\cos\phi + \phi \sin\phi_s) = constant \; , \tag{1}$$

where ΔE and ϕ are the energy difference and phase from the ideal particle of charge e, v is the speed of the particle, c is the speed of light, E_s is the synchronous energy, ϕ_s the synchronous phase, V the peak RF voltage, ω_{rf} the RF angular frequency, τ the transit time, and η the slip factor. This equation describes the trajectory of a particle in longitudinal phase space. For phases near the synchronous phase, $\phi = \phi_s + \Delta\phi$, with $\Delta\phi \ll 1$, the above can be re-written as

$$\Delta E^2 + \frac{2\beta^2 E_s eV}{2\pi h \eta}\left[\cos\phi_s \cos\Delta\phi - \sin\phi_s \sin\Delta\phi + (\phi_s + \Delta\phi)\sin\phi_s\right] = constant, \tag{2}$$

where we use the fact that $\omega_{rf}\tau = 2\pi h$, with h the harmonic number of the RF system, and $\beta \equiv v/c$. Noting that $\sin\Delta\phi \approx \Delta\phi$ and $\cos\Delta\phi \approx 1 - \frac{1}{2}\Delta\phi^2$, Eq. 2 reduces to

$$\Delta E^2 - \frac{\beta^2 E_s eV \cos\phi_s}{2\pi h \eta}\;\Delta\phi^2 = constant. \tag{3}$$

Finally, we convert to ΔE-Δt phase space, where $\Delta t = \Delta\phi/\omega_{rf} = \Delta\phi/(2\pi h f_0)$, with f_0 the revolution frequency. Since energy and time are canonical variables, the area (emittance) in phase space bounded by a single particle's trajectory is an adiabatic invariant. The resulting phase space trajectory of a particle is thus given by

$$\Delta E^2 - \frac{\beta^2 E_s eV \cos\phi_s}{2\pi h \eta}(2\pi h f_0)^2\;\Delta t^2 = constant. \tag{4}$$

The minus sign above is fine, since for stable motion the choice of $\cos\phi_s$ is made to ensure that the product $\eta\cos\phi_s$ is negative. Thus, Eq. 4 is the equation of an ellipse. To evaluate the constant, we look at the specific trajectory within which lies 95% of the particles in the bunch. It is easy to see that if we scale the axes until the trajectories are circular, and note that the distribution has been assumed to be Gaussian (and stationary in time), then the radius a which contains 95% of the particles will be given by

$$\frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^a e^{-r^2/2\sigma^2} r dr d\theta = \int_0^a e^{-r^2/2\sigma^2} \ \frac{rdr}{\sigma^2} = \int_0^{a^2/2\sigma^2} e^{-u}du = 1 - e^{-a^2/2\sigma^2} = 0.95\ . \tag{5}$$

$$\Longrightarrow a^2/2\sigma^2 = -\ln(0.05) \Longrightarrow a \approx \sqrt{6}\,\sigma$$

Thus, the equation of the 95% contour is

$$\Delta E^2 - \frac{\beta^2 E_s eV\cos\phi_s}{2\pi h\eta}(2\pi h f_0)^2\ \Delta t^2 = 6\sigma_E^2 \tag{6}$$

where σ_E is the rms of the energy distribution about the synchronous energy. Collecting terms into a more familiar form,

$$\frac{\Delta E^2}{6\sigma_E^2} - \frac{\beta^2 E_s eV\cos\phi_s}{12\pi h\eta\sigma_E^2}(2\pi h f_0)^2\ \Delta t^2 = 1\ . \tag{7}$$

The area S of this ellipse is called the **95% longitudinal emittance** and has units of eV-sec:

$$\begin{aligned} S &= \frac{3}{hf_0}\sqrt{\frac{-2\pi h\eta E_s^3}{\beta^2 eV\cos\phi_s}}\left(\frac{\sigma_E}{E_s}\right)^2 && (8)\\ &= \frac{3}{hf_0}\sqrt{\frac{-2\pi h\eta E_s^3\beta^2}{eV\cos\phi_s}}\left(\frac{\sigma_p}{p}\right)^2 && (9)\end{aligned}$$

where σ_p is the rms of the momentum distribution about the central momentum p, and we have used the fact that $\Delta p/p = (1/\beta^2)(\Delta E/E_s)$.

Given a specific 95% longitudinal emittance and the necessary RF system parameters, etc., the momentum spread of the beam is given by

$$\frac{\sigma_p}{p} = \left(-\frac{heV\cos\phi_s S^2 f_0^2}{18\pi\eta E_s^3\beta^2}\right)^{1/4}\ . \tag{10}$$

So, we have the situation depicted in the figure below. The area inside the separatrix is called the **bucket area** and, for a stationary (non-accelerating) bucket, is given by

$$\mathcal{A} = \frac{8C}{\pi hc}\sqrt{\frac{eV\ E_s}{2\pi h|\eta|}} \tag{11}$$

where C is the circumference of the accelerator.

For typical Main Injector parameters at 120 GeV,

$$\begin{aligned} \mathcal{A} &= \frac{8\cdot 3.3\times 10^3\mathrm{m}}{\pi(588)(3\times 10^8\mathrm{m/sec})}\sqrt{\frac{(3.5\times 10^6\mathrm{eV})\ (120\times 10^9\mathrm{eV})}{2\pi(588)(1/20)^2}} && (12)\\ &= 10\ \ \mathrm{eV}\cdot\mathrm{sec.} && (13)\end{aligned}$$

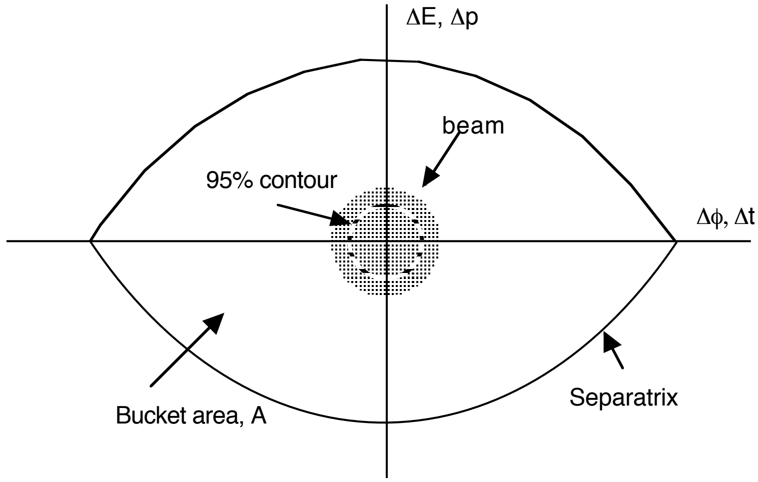

Figure 1: Stationary bucket with beam emittance indicated.

Likewise, for a longitudinal emittance of $S = 1$ eV·sec ($\ll \mathcal{A}$) the momentum spread is

$$\frac{\sigma_p}{p} = \left(\frac{(588)(3.5\times10^6\mathrm{eV})(1\mathrm{eV}\cdot\mathrm{sec})^2(91\mathrm{kHz})^2}{18\pi(1/20)^2(120\times10^9\mathrm{eV})^3}\right)^{1/4} \tag{14}$$

$$= 5\times10^{-4}\ . \tag{15}$$

We see that an rms relative momentum spread of 10^{-3} in the Main Injector at 120 GeV would correspond to a 95% longitudinal emittance of about 4 eV·sec.

There is often confusion concerning the 95% longitudinal emittance and the 95% momentum spread or energy spread. If the energy distribution is Gaussian, then $\pm2\sigma_E$ will contain ~95% of the particles. If one follows a $2\sigma_E$ particle in phase space and finds the corresponding $2\sigma_t$ edge in the parameter Δt, then the area bounded by this ellipse is $4\pi\sigma_E\sigma_t$. While $\pm2\sigma_E$ contains 95% of the particles, and $\pm2\sigma_t$ also contains 95% of the particles, the curve in phase space which bounds the area $4\pi\sigma_E\sigma_t$ does NOT contain 95% of the particles. In fact, it contains only 86.5% of the particles. The 95% emittance is actually 50% larger than this number. That is,

$$S = 6\pi\sigma_E\sigma_t = 6\pi\beta c\sigma_p\sigma_t\ \ .$$

References

[1] D. A. Edwards and M. J. Syphers, *An Introduction to the Physics of High Energy Accelerators*, J. Wiley, & Sons, Inc., New York (1993).