Tuesday

- Wish to cover (lots!):
  - Matrix formalism, and Strong Focusing
- Hill's Equation
  - Analytical Solutions to Hill's Equation
- Courant-Snyder parameters (beta function, ...)
  - motivation and meaning
  - computation

Back to Transverse Motion…
Piecewise Method of Solution

- Hill's Equation $$x'' + Kx = 0$$
  - Though $K(s)$ changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, sector mag, quad, edge, ...)

drift

  - K = 0: $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

Quad, Gradient Magnet, edge, ...

  - K > 0: $$x(s) = x_0 \cos(\sqrt{K}\; s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\; s)$$
  - K < 0: $$x(s) = x_0 \cosh(\sqrt{|K|} s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|} s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L
- $K = 0$: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$
- $K > 0$: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$
- $K < 0$: $$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

"Thin Lens" Quadrupole

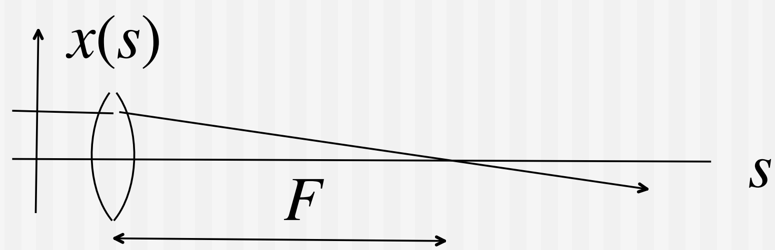
- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics
- Take limit as L --> 0, while KL remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

  - (similarly, for defocusing quadrupole)

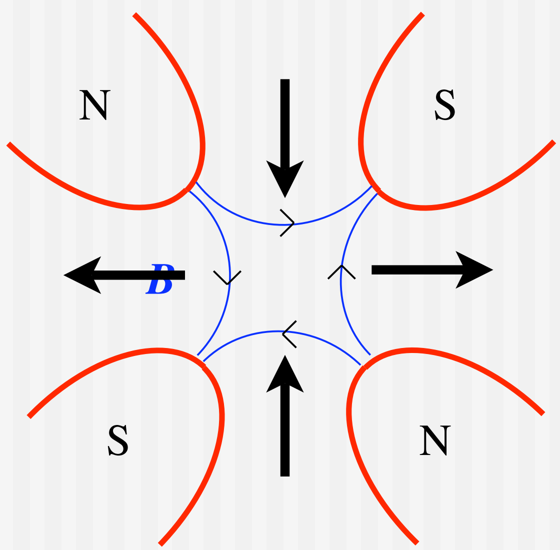
- Valid approx., if F >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$



Quadrupole Field

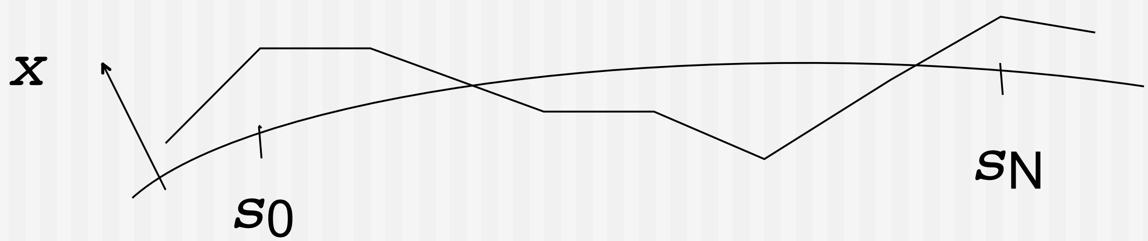
- Note: A quadrupole magnet will focus in one plane, and defocus in the other



Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed *via* matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Review of Matrices

- Operation of matrix on a vector
- Addition of matrices
- Multiplication of matrices
- Determinant of a matrix
- Trace of a matrix
- Eigen-values and eigen-vectors

Lens Systems

- Singlet
- Doublet
- Triplet

- Matrix descriptions of thin lens systems -- ray tracing

Stability Criterion

- For single pass through a system of elements, above treatment may be enough to describe the system. Suppose the "system" is a synchrotron -- how to show that the motion is stable for many (infinite?) revolutions? (24 hrs x 50K rev/sec = …)
- Look at matrix describing motion for one revolution:

$$M = M_N M_{N-1} \cdots M_2 M_1$$

- We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Stability Criterion

v = eigenvector
λ = eigenvalue

$$X_k = M^k X_0 = M^k(AV_1 + BV_2) = A\lambda_1^k V_1 + B\lambda_2^k V_2$$

$$\det M = 1 = \lambda_1\lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ real, gives oscillatory solutions…

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

$$\begin{aligned}
&\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\
&\lambda^2 - tr M\lambda + 1 = 0 \\
&\lambda + 1/\lambda = tr M \\
&e^{i\mu} + e^{-i\mu} = 2\cos\mu = tr M
\end{aligned}$$

$$\boxed{\begin{array}{c}\text{So, } \mu \text{ real (stability)} \\ \rightarrow |tr M| < 2\end{array}}$$

Discovery of Strong Focusing*

- Consider weak focusing system discussed earlier, made up of 2*N* identical gradient magnets. Take every other magnet, turn it around so that the wedge opens inward, and reverse its current.
- Then all magnets have same bend field (in same direction) on the ideal trajectory, but every other magnet has its gradient (*K*) with reversed sign. We now have *N* "cells" of +*K* and -*K.*
- In one degree-of-freedom (vertical, say), each cell has matrix:

$$
\begin{aligned}
M_c &= \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\sqrt{K}L)\cosh(\sqrt{K}L) + \sin(\sqrt{K}L)\sinh(\sqrt{K}L) & \cdots \\ \cdots & \cos(\sqrt{K}L)\cosh(\sqrt{K}L) - \sin(\sqrt{K}L)\sinh(\sqrt{K}L) \end{pmatrix}
\end{aligned}
$$

from which

$$tr M = 2\cos(\sqrt{K}L)\cosh(\sqrt{K}L)$$

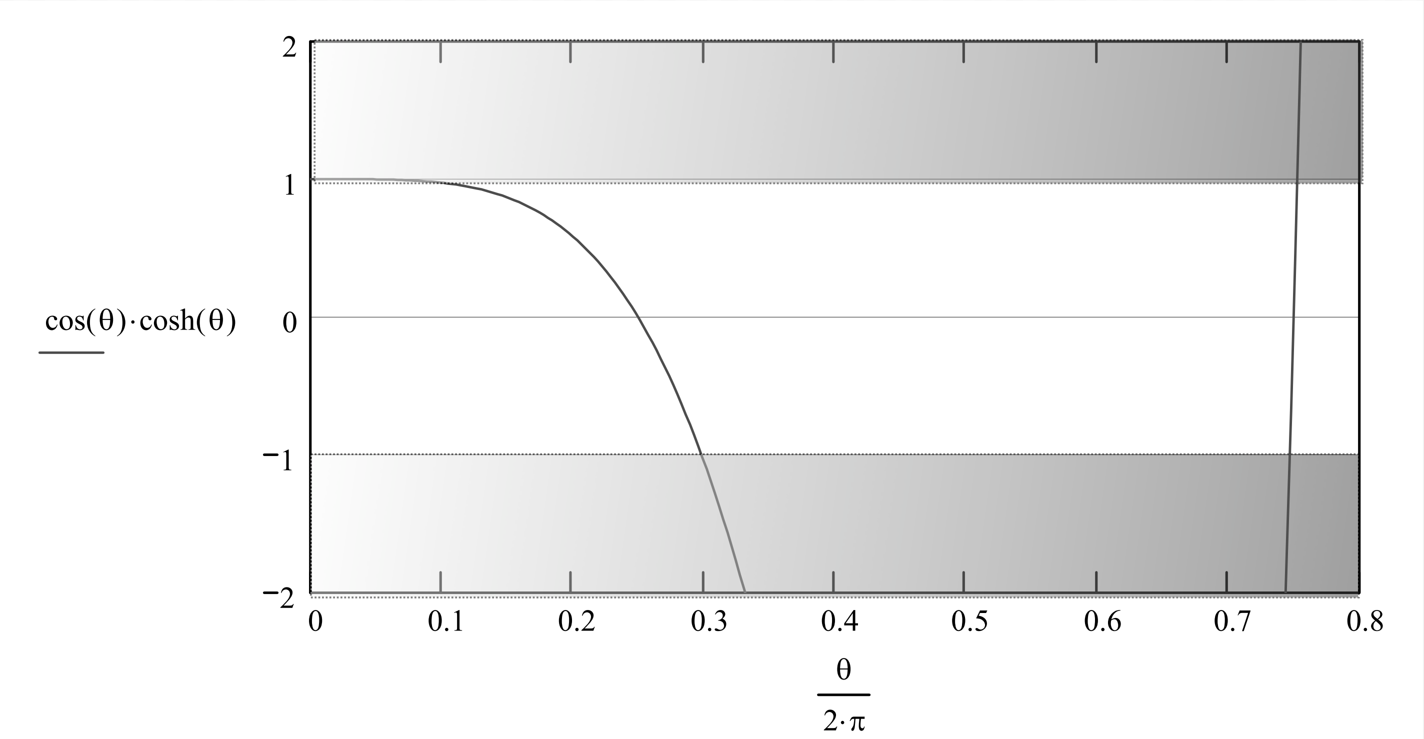
So, we need

$$|\cos(\sqrt{K}L)\cosh(\sqrt{K}L)| < 1$$

Here, $K = |B'|/B\rho$

*Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950.

The Strong vs. The Weak…



So, could choose $KL^2 \approx [(0.2)2\pi]^2 = 1.58$, say.

$\rightarrow KL^2 = (B'/B\rho)L^2 = (B'\rho/B)(L/\rho)^2 = |n|\ \theta_0^2 = |n|\ (2\pi/2N)^2 = 1.58$

for example, say $N \sim 25$; then $|n| \sim 100 \gg 1$ (*STRONG* focusing!)

(Note: for Weak Foc. accel., typically $n \sim 2/3 \longrightarrow KL^2 \sim 0.01$!)

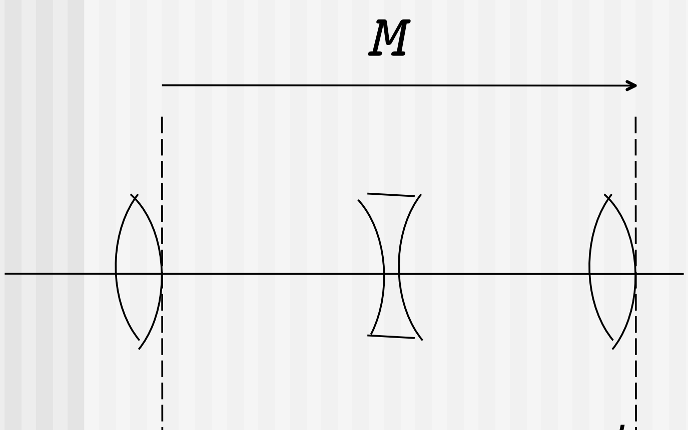
Alternating Gradients

- So, now that we can accommodate very strong field gradients, and alternate them over short distances, the extent of radial and vertical excursions becomes decoupled from the orbital radius of the accelerator.

- The announcement of the AG concept came in 1952, and was immediately applied at Cornell in a 1 GeV electron synchrotron being constructed (Wilson, *et al.*), the world's highest energy at the time. This eventually led to the design and construction of the PS at CERN (1958) and the AGS at Brookhaven National Lab (1960), increasing particle energies to the 30 GeV range. Strong Focusing has been at the heart of every forefront accelerator ever since.

Application to FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $tr M = 2 - L^2/F^2$ and thus, for stability,

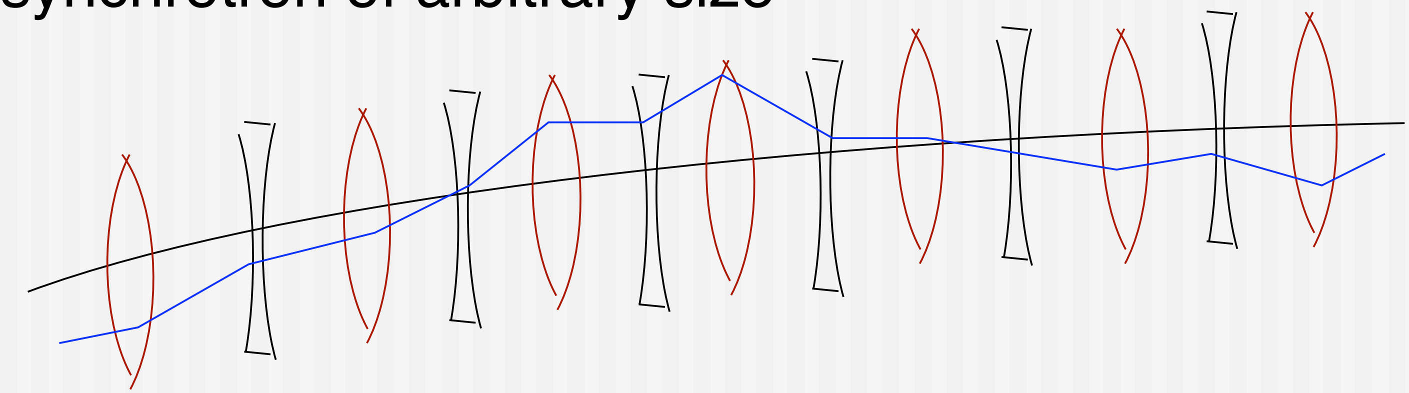
$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

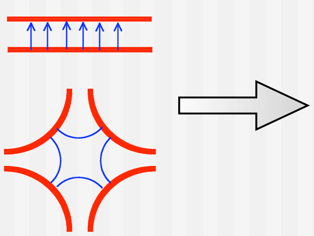
$$F > L/2$$

Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing fields, then in principal can make a synchrotron of arbitrary size

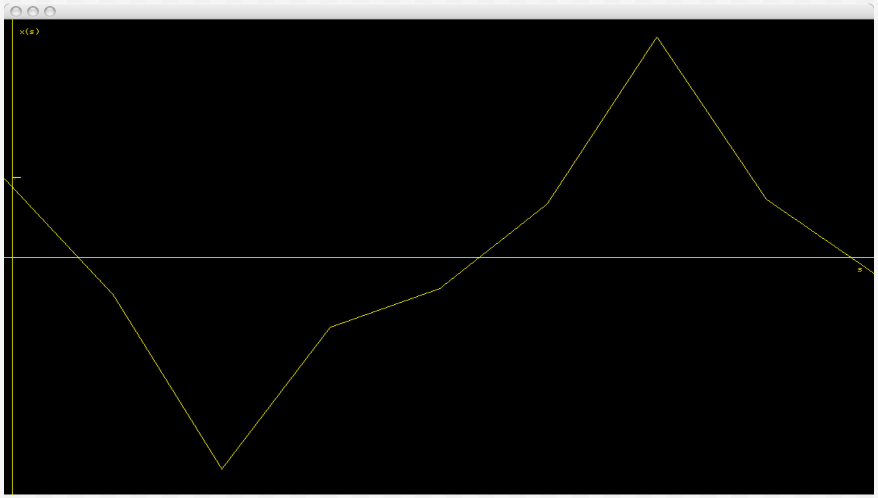


- Can “separate” the bending and focusing “functions”
- First synchrotron to use alternating gradient “thin lenses” + dipole magnets:
  - Fermilab Main Ring

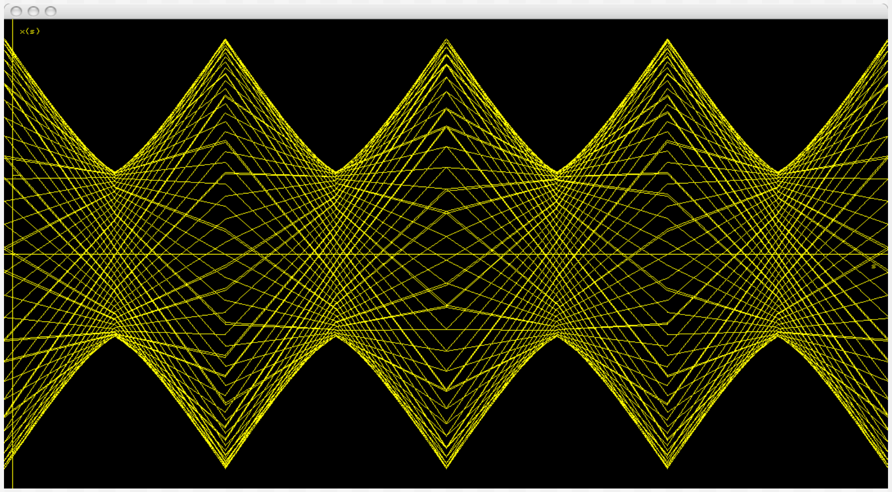


The Notion of an Amplitude Function…

- Track a single particle through a system of FODO cells



- Repeat, representing multiple passages around a synchrotron

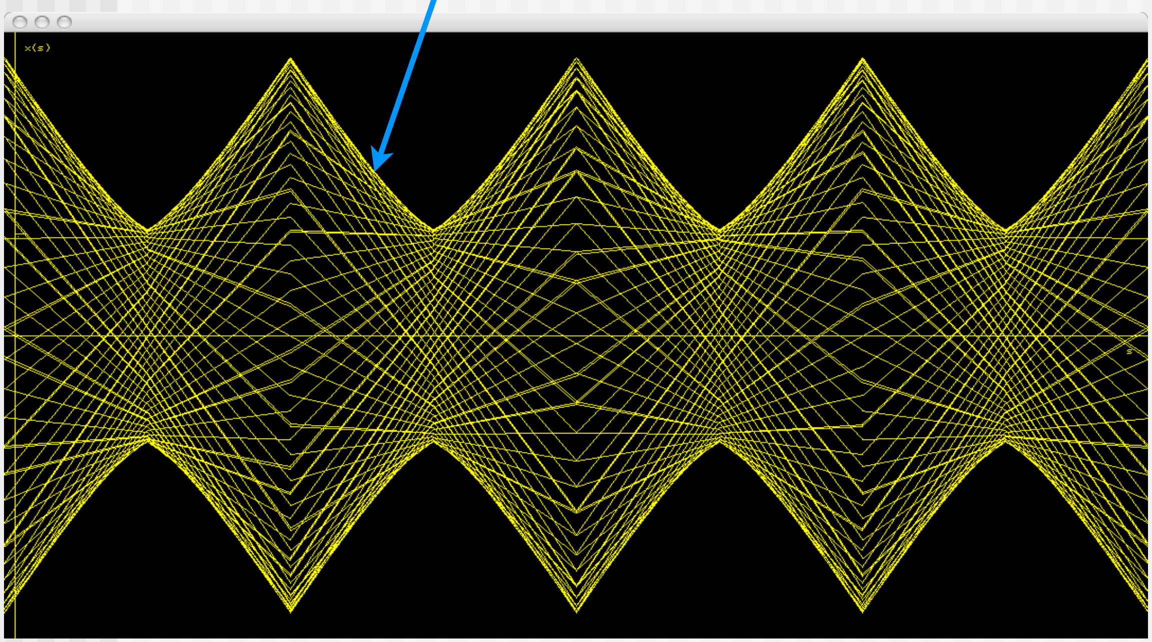


Can we describe the maximum amplitude of particle excursions in analytical form?

of course!

Pushing the "Envelope"

Envelope described by an
"amplitude function"



- We saw, for a FODO system, that the motion of a single particle is contained within an "envelope"
- Wish to determine its functional form, and the rate at which the phase of the oscillatory motion develops
- Decouple motion of individual particle from intrinsic properties of the accelerator design

Hill's Equation -- Analytical Solution

- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, s.
- So, assume solution is sinusoidal, with a phase which advances as a function of location s; also assume amplitude is modulated by a function which also depends upon s:

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

- Plug into Hill's Equation...

Analytical Solution (cont'd)

$$
\begin{aligned}
x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\
x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\
x'' &= \ldots
\end{aligned}
$$

Plug into Hill's Equation, and collect terms...

$$
\begin{aligned}
x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\
&\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0
\end{aligned}
$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, the contents of each [ ] must be zero simultaneously.

Analytical Solution (cont'd)

- Thus, we must have ...

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

and

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

Note: the phase advance is an observable quantity. So, while could choose different value of $const$, then β would just be scaled accordingly; so, we can choose $const$ = 1.

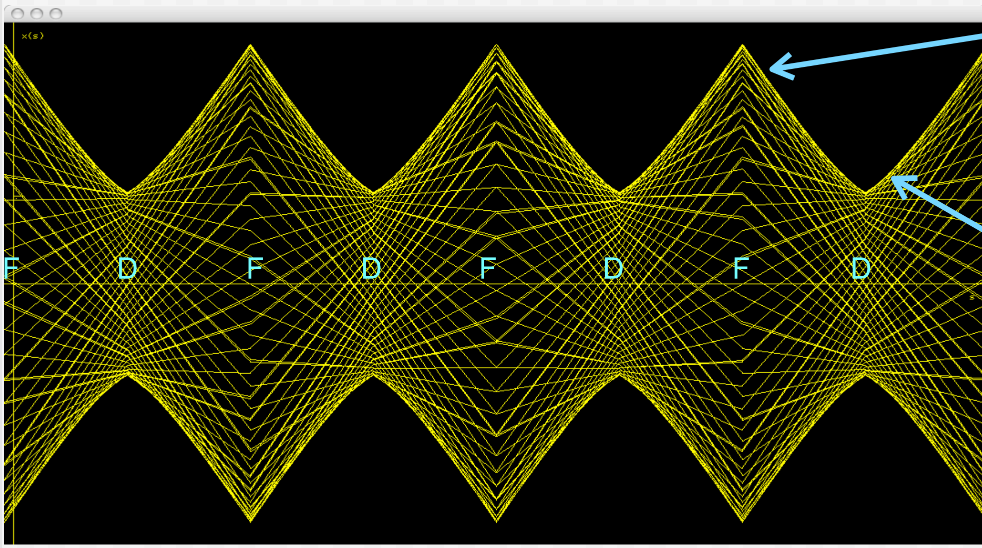
The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Differential equation that the amplitude function must obey

Some Comments

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But it also is a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., $K(s)$) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes *must* be tied together.

The Amplitude Function, β



Higher β --
smaller phase advance
larger beam size

Lower β --
greater phase advance
smaller beam size

- Since the amplitude function is a wavelength, it will have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant A must have units of m$^{1/2}$, and it must be numerically small. More on this subject later ...

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.
(in regions where K is either zero or constant)

Piecewise Solutions

- $K = 0$:

$$\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

Parabola!

  - since $\beta > 0$, then from original diff. eq.: $2\beta\beta'' - (\beta')^2 = 4$
  - Therefore, parabola is always concave up $\beta'' > 0$

- $K > 0$, $K < 0$:

$$\beta(s) \sim \sin / \cos \quad \text{or} \quad \sinh / \cosh \quad + \quad const$$

Homework for Wednesday

- Problem Set 2 -- Numbers 1, 2, 6, and 7

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}$$

- Collectively, β, α, γ are called the Courant-Snyder Parameters (sometimes called "Twiss" or "lattice" parameters)

The Transport Matrix

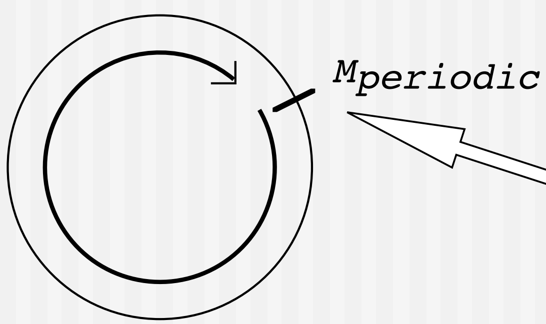
- We can write: $x(s) = a\sqrt{\beta}\sin\psi + b\sqrt{\beta}\cos\psi$
- Solve for a and b in terms of initial conditions and write in matrix form
  - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Periodic Solutions

- Within a system made up of periodic sections it is natural to want the beam envelope to have the same periodicity.
- Taking the previous matrix to be that of a periodic section, and demanding the C-S parameters be periodic yields...

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$



values of β, α above correspond to one particular point in the accelerator

Periodicity and the "Tune"

- We see from above that matrix of a periodic section (which, for example, could be an entire synchrotron!) has a Trace which is

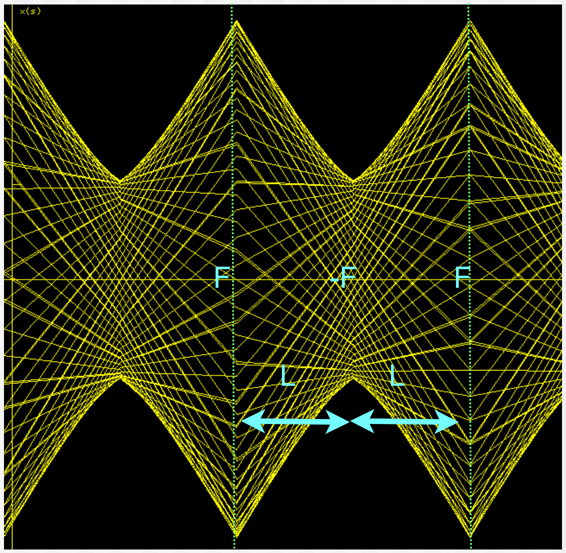
$$trace(M_{periodic}) = 2\cos\Delta\psi$$

- If the matrix *does* represent an entire synchrotron, then the total phase advance is just $2\boldsymbol{\pi}$ x the tune:

$$\Delta\psi = 2\pi\nu = \oint \frac{ds}{\beta(s)}$$

Computation of Courant-Snyder Parameters

- As an an example, consider a FODO system



$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Thus, use above matrix to compute functions at exit of the F quad..

FODO Cell

- From the matrix:

Here, μ is phase advance through one periodic cell

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$traceM = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1+\sin\mu/2}{1-\sin\mu/2}} \qquad \alpha = \frac{a-d}{2\sin\mu} = \sqrt{\frac{1+\sin\mu/2}{1-\sin\mu/2}}$$

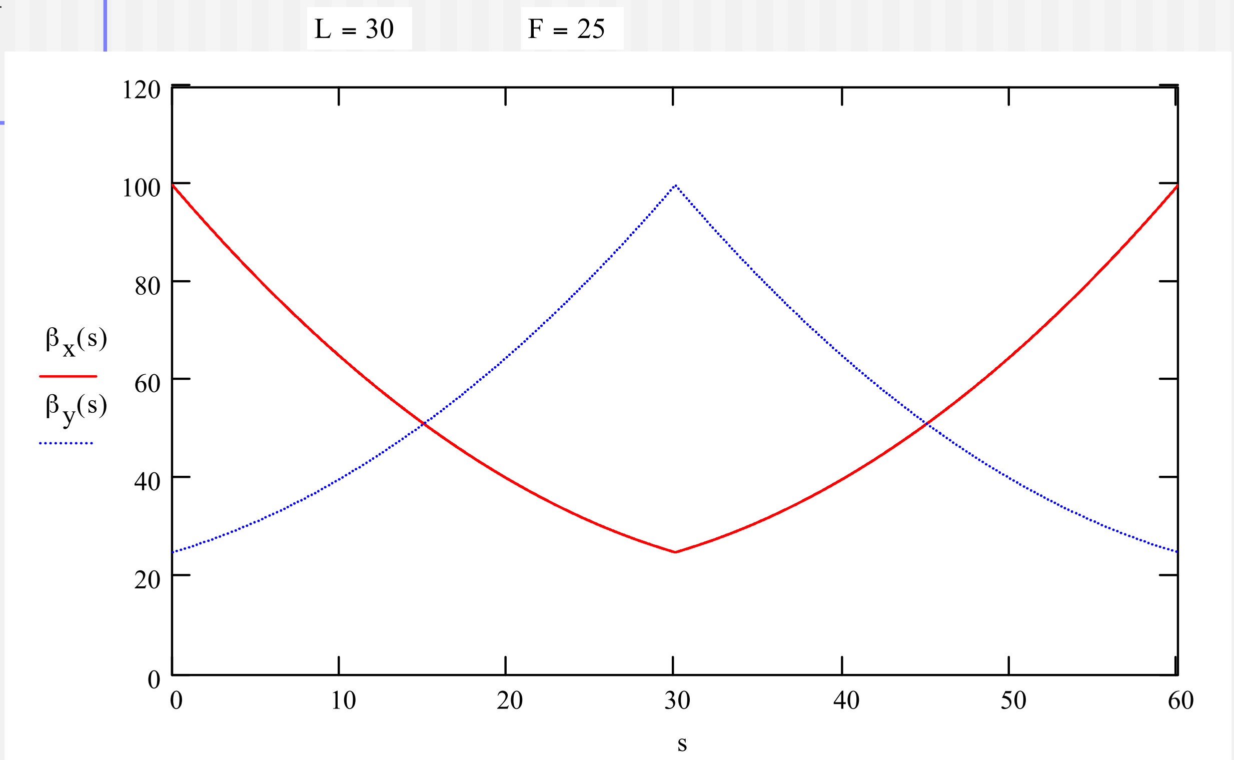
- If go from D quad to D quad, get
  - at exit:

$$\beta = 2F\sqrt{\frac{1-\sin\mu/2}{1+\sin\mu/2}}, \quad \alpha = \sqrt{\frac{1-\sin\mu/2}{1+\sin\mu/2}}$$

Periodic FODO Cell Functions

- Tevatron Cell

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25\ \mathrm{m})\sqrt{1.6/0.4} = 100\ \mathrm{m}$$
$$\beta_{min} = 2(25\ \mathrm{m})\sqrt{0.4/1.6} = 25\ \mathrm{m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$



Propagation of Courant-Snyder Parameters

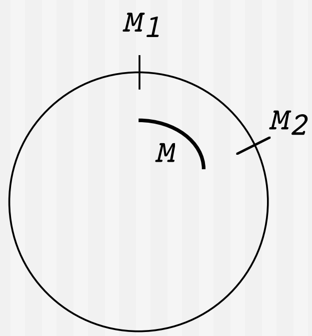
- We note that can write periodic matrix corresponding to location *s* as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi \\
&= I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}
\end{aligned}
$$

- where $J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$ $\quad detJ = 1, \quad trace(J) = 0; \quad J^2 = -I$

Tracking β, α, γ ...

- Let M_1 and M_2 be the "periodic" matrices at two points, and M propagates the motion between them. Then,



$$M_2 = M\ M_1\ M^{-1}$$

$(M_1, M_2$ are "once around")

- Or, equivalently, $$\boxed{J_2 = M\ J_1\ M^{-1}}$$
  - So, if know parameters (*i.e.*, J) at one point, can find them at another point if given the matrix for motion in between

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase and C-S parameters along a beam line

Simple Examples

- Propagation through a Drift

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
\Longrightarrow \quad & \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right) \\
& \beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2 \\
& \alpha = \alpha_0 - \gamma_0 L \\
& \gamma = \gamma_0
\end{aligned}
$$

- Propagation through a Thin Lens

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \\
\Longrightarrow \quad & \Delta\psi = 0 \\
& \beta = \beta_0 \\
& \alpha = \alpha_0 + \beta_0/F \\
& \gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2
\end{aligned}
$$

Choice of Initial Conditions

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the periodic solution for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- Will discuss optical "mismatches" and their implications in future talks

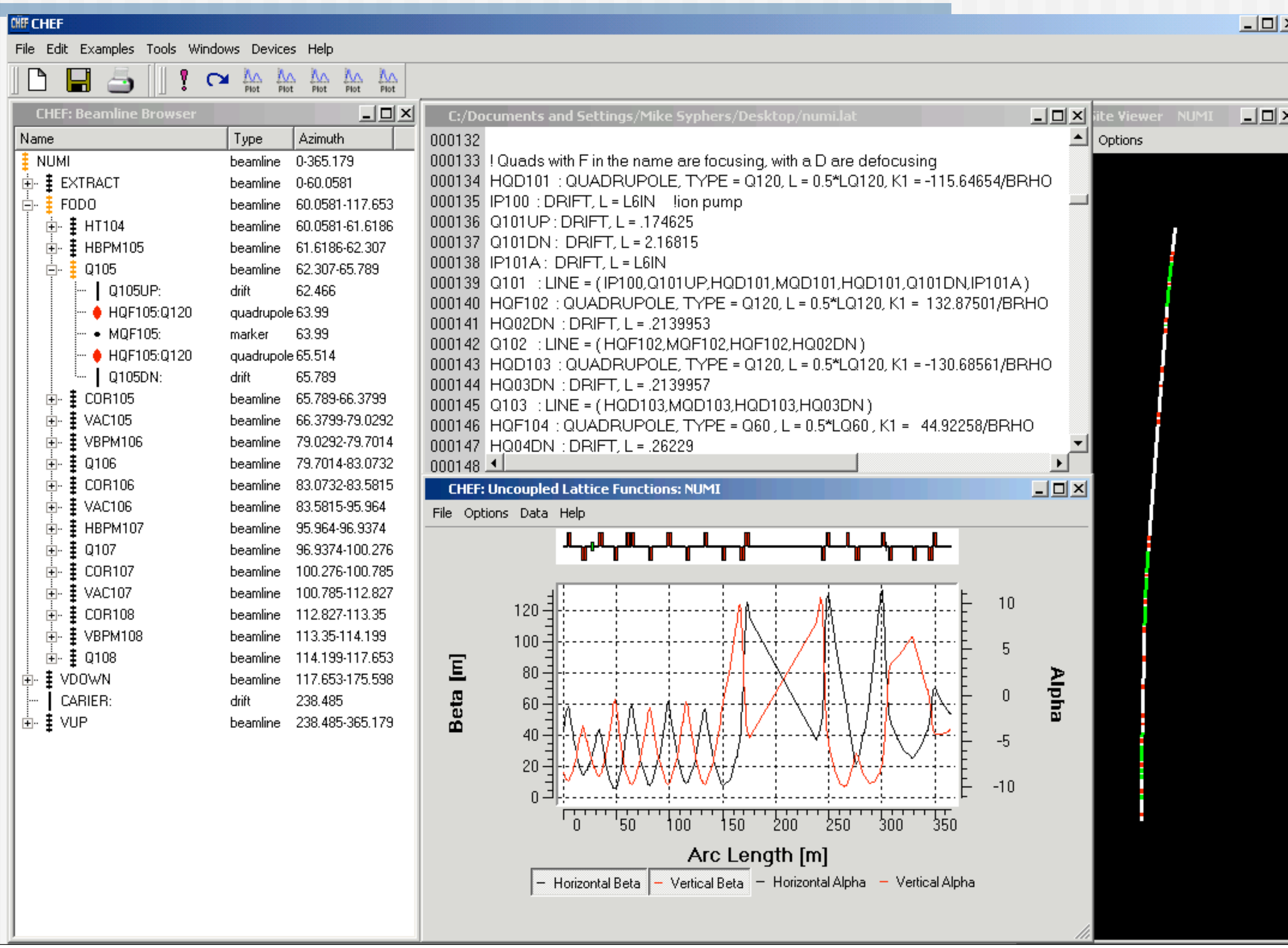
Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT
  - SYNCH
  - MAD
  - CHEF
  - many more ...

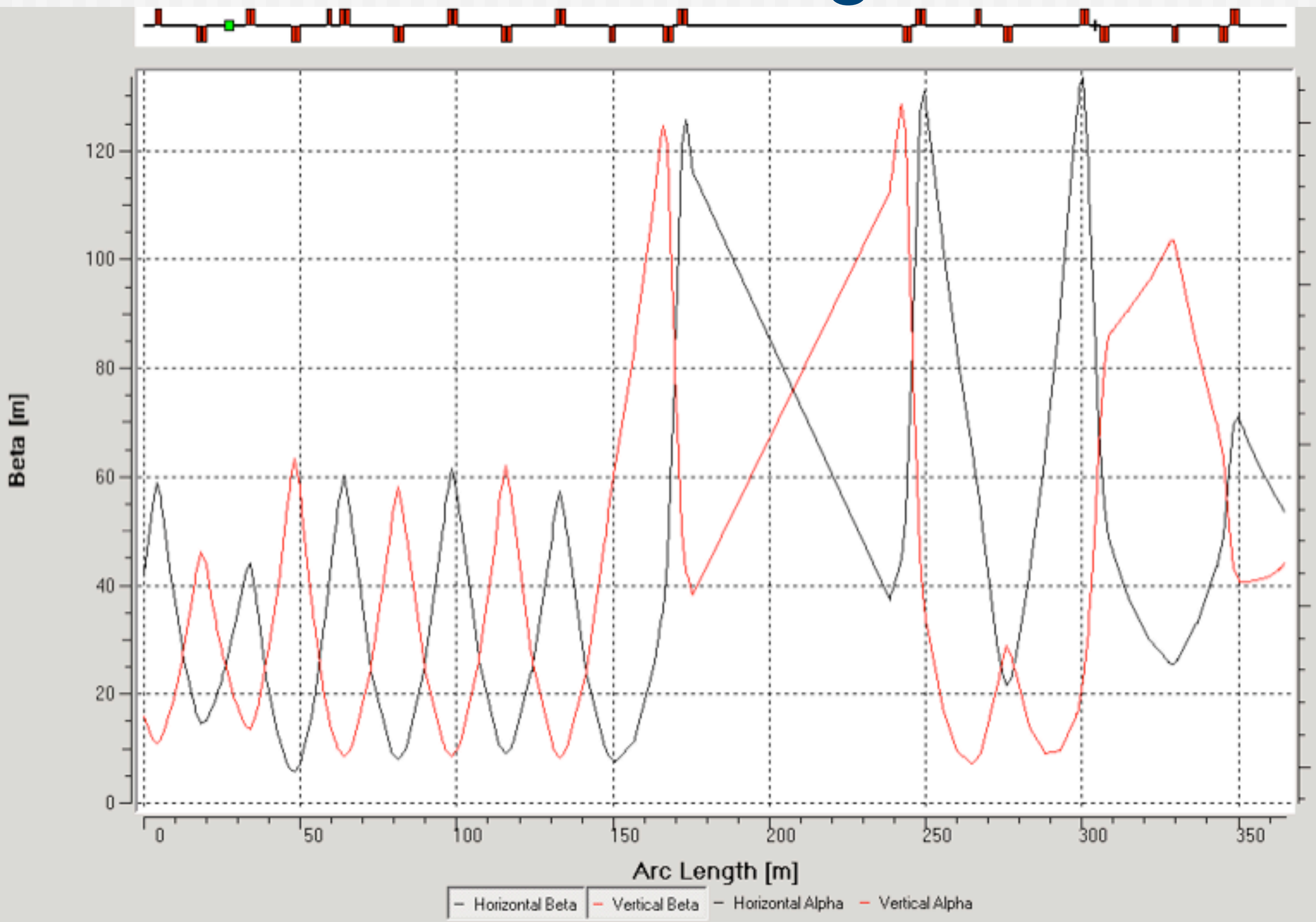
An Example -- NuMI Beam Line

- Using CHEF

(Michelotti, Ostiguy)



NuMI Beam Line using CHEF



Homework for Wednesday

- Problem Set 2 -- Numbers 1, 2, 6, and 7