# Wednesday

- Today:
  - Transverse Phase Space
  - Emittance (single particle, and beam)
  - Off-Momentum considerations
    - Dispersion
    - Path length

# Overview

- Found analytical solution to Hill's Equation:

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- So far, discussed amplitude function, β
- What about *A*?
  - Given β(s), how big is the beam at a particular location? mm? cm? m?
  - If perturb the beam's trajectory, how much will it move downstream?
- Single particle behavior *vs*. a "beam"

# Betatron Oscillation Amplitude

- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a *betatron*)
- Given

$$
\begin{aligned}
x &= a\sqrt{\beta}\sin\psi + b\sqrt{\beta}\cos\psi \\
x' &= \frac{1}{\sqrt{\beta}}\left([b - a\alpha]\cos\psi - [a + b\alpha]\sin\psi\right) \\
&\downarrow \\
a &= \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}}
\end{aligned}
$$

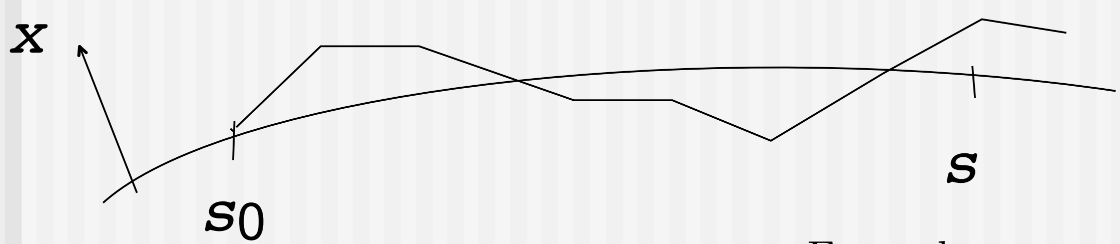
$$
\boxed{\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi\right]}
$$

$$
\text{amplitude: } A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}
$$

# Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = x'_0 = \Delta\theta$
- Then, downstream, we have

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\,\sin[\psi(s) - \psi_0]$$




Example:
Suppose $\Delta\theta = 0.1$ mrad, $\beta_0 = 49$ m, $\beta(s) = 64$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 2.8$ mm.

# Courant-Snyder Invariant

- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi}
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

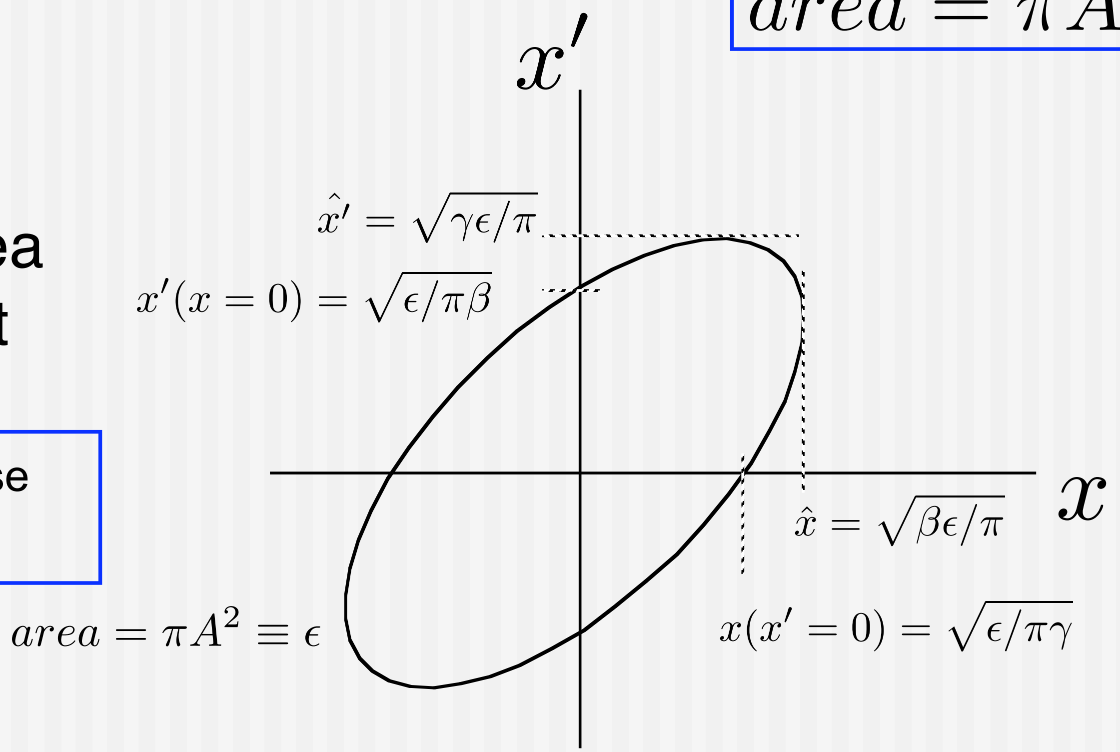
While C-S parameters evolve along the beam line, the combination above remains constant.

# Properties of the Phase Space Ellipse

- The eqn. for the C-S invariant is that of an ellipse. If compute the area of the ellipse, find that

$$\boxed{area = \pi A^2}$$

- i.e., while the ellipse changes shape along the beam line, its area remains constant

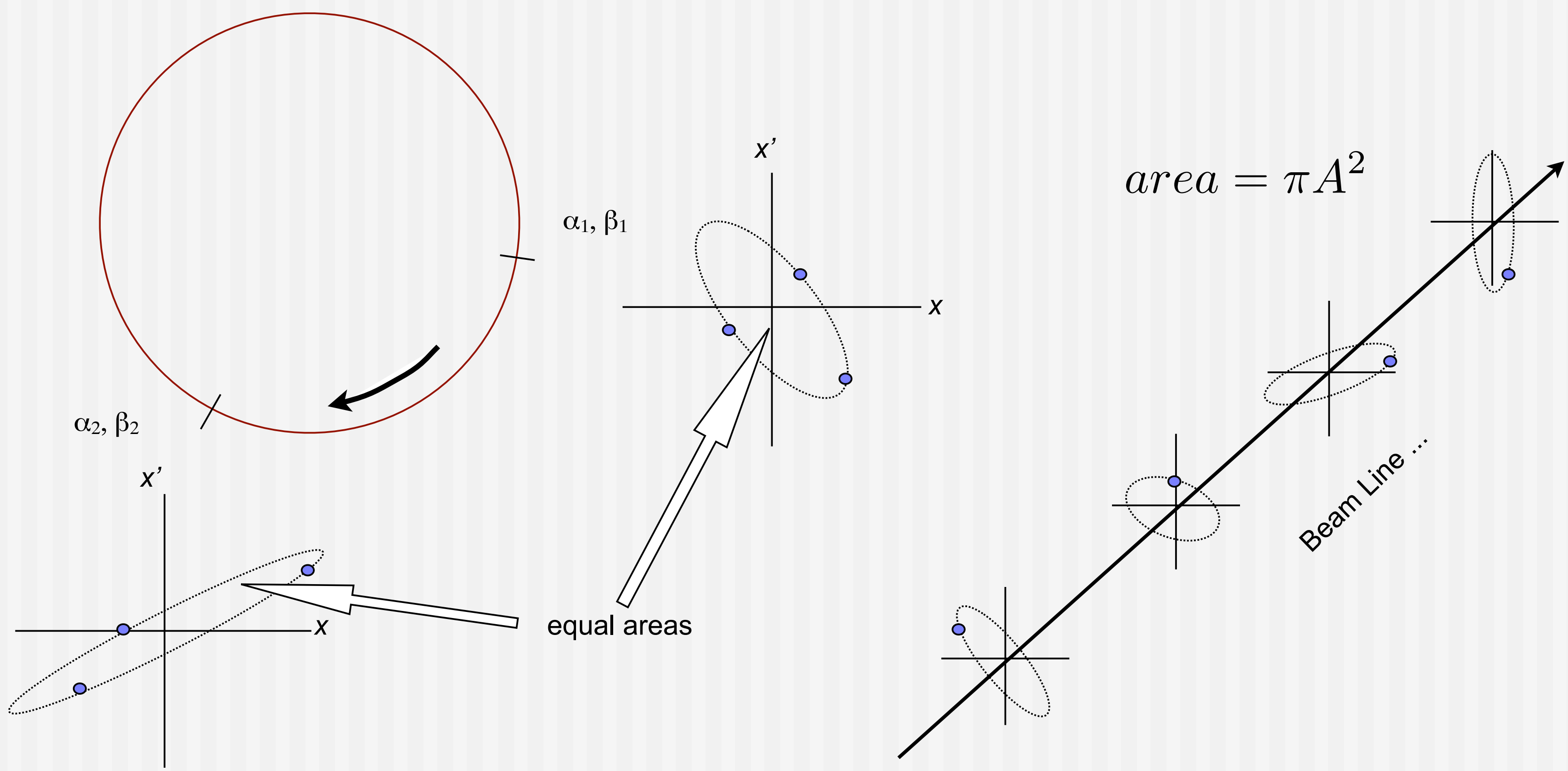



**_Emittance_** = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$

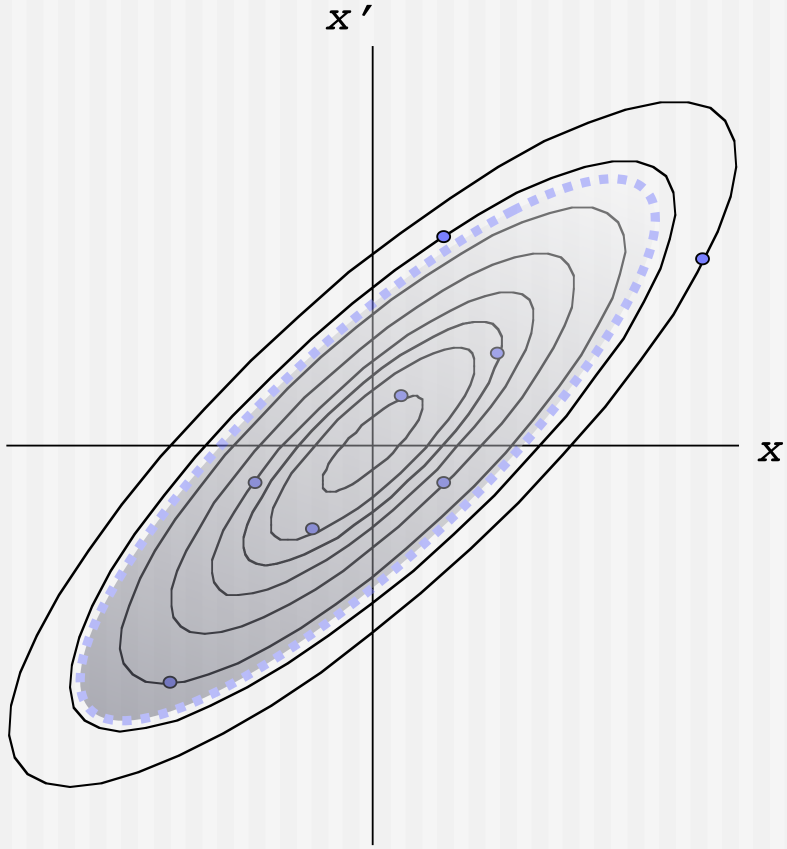
# Motion in Phase Space

- Follow phase space trajectory...



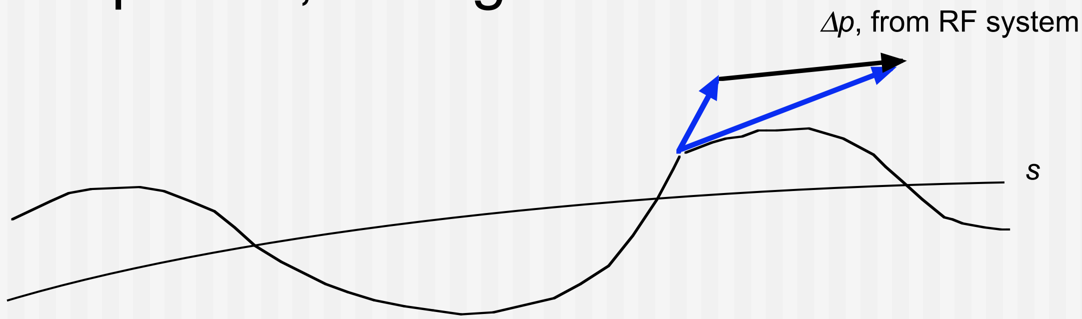
# Beam Emittance

- Phase space area which contains a certain fraction of the beam particles
- Popular Choices:
  - 95%
  - 39%
  - 15%
  - ...more on this subject coming up...



# Adiabatic Damping from Acceleration

- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the $s$-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.




- The coordinates $x$–$x'$ are not canonical conjugates, but $x$–$p_x$ are; thus, the area of a trajectory in $x$–$p_x$ phase space is invariant for adiabatic changes to the system.

# Normalized Beam Emittance

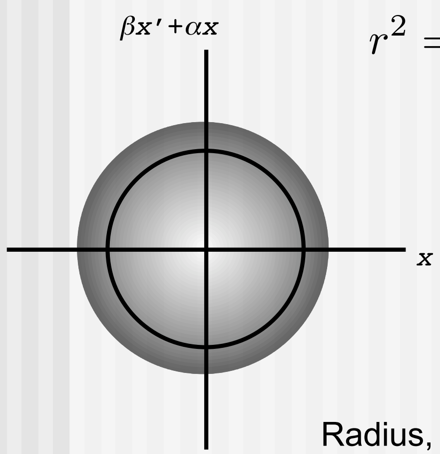
- Hence, as particles are accelerated, the area in $x$–$x'$ phase space is not preserved, while area in $x$–$p_x$ is preserved. Thus, we define a "normalized" beam emittance, as

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)$$

- In principle, the normalized beam emittance should be preserved during acceleration, and hence along the chain of accelerators (at FNAL: Linac, Booster, Main Injector, etc.). Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.

# Gaussian Beam in a Periodic System

- Imagine a synchrotron in which the transverse distribution of circulating particles has reached an equilibrium with a Gaussian profile in transverse coordinate *x* with zero mean and standard deviation $\sigma$.
  - The distribution can be described as follows:




$$r^2 = x^2 + (\beta x' + \alpha x)^2$$

$$\rho(r,\theta) r dr d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r \, dr d\theta$$

$$\int_0^{2\pi} \int_0^a \rho \, r dr d\theta = f$$

Radius, *a*, containing fraction, *f*, of particles, corresponding to phase space area with emittance, ε:

$$a^2 = -2\sigma^2 \ln(1-f) = \epsilon\beta/\pi$$

# Gaussian Emittance

- So, the normalized emittance that contains a fraction *f* of a Gaussian beam is:

$$\epsilon_N = \frac{-2\pi \ln(1-f)\sigma^2(s)}{\beta(s)}(\beta\gamma)$$

Lorentz!

- Common values of *f* :

| f | $\epsilon_N/(\beta\gamma)$ |
|---|---|
| 95% | $6\pi\sigma^2/\beta$ |
| 86.5% | $4\pi\sigma^2/\beta$ |
| 39% | $\pi\sigma^2/\beta$ |
| 15% | $\sigma^2/\beta$ |

*(rms emittance)*

# Emittance Measurements

- Typical practice, in a synchrotron, is to measure rms beam size (assumed Gaussian), at a location where $\beta$ is presumed to be known, and thus emittance can be deduced.
- While Gaussian description is often good approximation of the distribution, not necessarily true. Also possible to define the emittance in terms of *2nd* moments of the (arbitrary) distribution.

# Emittance in Terms of Moments

- For each particle, $x = A\sqrt{\beta}\sin\psi \qquad x' = \frac{A}{\sqrt{\beta}}(\cos\psi - \alpha\sin\psi)$
- Average over the (stationary) distribution...

$$x^2 = A^2\beta\sin^2\psi \qquad x'^2 = \frac{A^2}{\beta}(\cos^2\psi + \alpha^2\sin^2\psi - \alpha\sin 2\psi)$$

$$\langle x^2\rangle = \frac{1}{2}\langle A^2\rangle\beta \qquad \langle x'^2\rangle = \frac{\langle A^2\rangle}{2\beta}(1+\alpha^2) = \frac{1}{2}\langle A^2\rangle\gamma$$

and ...

$$\begin{aligned} xx' &= A^2(\frac{1}{2}\sin 2\psi - \alpha\sin^2\psi) \\ \langle xx'\rangle &= -\frac{1}{2}\langle A^2\rangle\alpha \end{aligned}$$

$\beta\gamma - \alpha^2 = 1$

From which the average of all particle emittances will be $\pi\langle A^2\rangle = 2\pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$

and the "normalized rms emittance" can be defined as: $\boxed{\epsilon_N = \pi(\beta\gamma)\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}}$

# TRANSPORT of Beam Moments

- For simplicity, define $\tilde{\epsilon} \equiv \frac{1}{2}\langle A^2 \rangle$ ; then,

$$\tilde{\epsilon} J = \begin{pmatrix} \tilde{\epsilon}\alpha & \tilde{\epsilon}\beta \\ -\tilde{\epsilon}\gamma & -\tilde{\epsilon}\alpha \end{pmatrix} = \begin{pmatrix} -\langle xx' \rangle & \langle x^2 \rangle \\ -\langle x'^2 \rangle & \langle xx' \rangle \end{pmatrix}$$

- Correlation Matrix:

$$\Sigma \equiv \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix} = -\tilde{\epsilon} J S, \quad where \;\; S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- then,

$$\begin{aligned} \Sigma_2 = -\tilde{\epsilon} J_2 S \quad &= \quad -\tilde{\epsilon} M J_1 M^{-1} S \\ &= \quad M(-\tilde{\epsilon} J_1 S) S^{-1} M^{-1} S \\ &= \quad M \Sigma_1 S^{-1} M^{-1} S \\ &= \quad M \Sigma_1 M^T \end{aligned}$$

Here, *M* is from point 1 to point 2 along the beam line

# Summary

- So, can look at propagation of amplitude function through beam line given matrices of individual elements. Beam size at a particular location determined by

$$x_{rms}(s) = \sqrt{\beta(s)\epsilon_N/\pi(\beta\gamma)}$$

- Or, given an initial particle distribution, can look at propagation of second moments (position, angle) given the same element matrices.
- Have neglected:
  - dispersion of trajectories due to momentum (coming up)
  - hor-ver coupling (typically zero, by design)

# Off-Momentum

- Equation of motion
- Piecewise/matrix approach to solution
- Periodic solution
- Example:  FODO cell with bending
- Dispersion in a chicane
- Total beam size, including dispersion
- Transition energy in a synchrotron

# Homework for Thursday

- Problem Set 2, No. 4
- Problem Set 3, Nos. 1, 4, and 5