Multipole feed-down

For the feed-down calculation the new coordinate system (u,v) is defined as:

$$u = x - x_0$$
$$v = y - y_0$$

Quadrupole

Normal

$$B_u = \frac{B_2}{a}(v + y_0) = \frac{B_2}{a}v + \frac{B_2 y_0}{a}$$
$$B_v = \frac{B_2}{a}(u + x_0) = \frac{B_2}{a}u + \frac{B_2 x_0}{a}$$

Skew

$$B_u = \frac{-A_2}{a}(u + x_0) = \frac{-A_2}{a}u + \frac{-A_2 x_0}{a}$$
$$B_v = \frac{A_2}{a}(v + y_0) = \frac{A_2}{a}v + \frac{A_2 y_0}{a}$$

Sextupole

Normal sextupole

$$B_u = \frac{B_3}{a^2}\left[2\cdot(u + x_0)\cdot(v + y_0)\right]$$
$$= \frac{B_3}{a^2}(2uv) + \frac{2y_0 B_3}{a^2}u + \frac{2x_0 B_3}{a^2}v + \frac{B_3}{a^2}(2x_0 y_0)$$
$$B_v = \frac{B_3}{a^2}\left[(u + x_0)^2 - (v + y_0)^2\right]$$
$$= \frac{B_3}{a^2}(u^2 - v^2) + \frac{2x_0 B_3}{a^2}u + \frac{-2y_0 B_3}{a^2}v + \frac{B_3}{a^2}(x_0^2 - y_0^2)$$

Multipole strength from sextupole feed-down

	Quadrupole	Dipole
Normal	$\frac{2x_0 B_3}{a^2}$	$\frac{B_3}{a^2}\left({x_0}^2 - {y_0}^2\right)$
Skew	$\frac{-2y_0 B_3}{a^2}$	$\frac{B_3}{a^2}(2x_0 y_0)$

Skew sextupole

$$B_u = \frac{-A_3}{a^2}\left[(u + x_0)^2 - (v + y_0)^2\right]$$
$$= \frac{-A_3}{a^2}(u^2 - v^2) + \frac{-2x_0 A_3}{a^2}u + \frac{2y_0 A_3}{a^2}v + \frac{-A_3}{a^2}(x_0^2 - y_0^2)$$

$$B_v = \frac{A_3}{a^2}\left[2\cdot(u+x_0)\cdot(v+y_0)\right]$$
$$= \frac{A_3}{a^2}(2uv) + \frac{2y_0A_3}{a^2}u + \frac{2x_0A_3}{a^2}v + \frac{A_3}{a^2}(2x_0y_0)$$

Multipoles from skew sextupole feed-down

	Quadrupole	Dipole
Normal	$\frac{2y_0A_3}{a^2}$	$\frac{A_3}{a^2}(2x_0y_0)$
Skew	$\frac{2x_0A_3}{a^2}$	$\frac{-A_3}{a^2}\left(x_0{}^2 - y_0{}^2\right)$

Octupole

Normal

$$B_u = \frac{B_4}{a^3}\left[3\cdot(u+x_0)^2(v+y_0)-(v+y_0)^3\right]$$
$$= \frac{B_4}{a^3}\left(3\cdot u^2v - v^3\right) + \frac{3x_0B_4}{a^3}(2uv) + \frac{3y_0B_4}{a^3}\left(u^2-v^2\right)$$
$$+\frac{6x_0y_0B_4}{a^3}u + \frac{3\cdot\left(x_0^2-y_0^2\right)B_4}{a^3}v + \frac{B_4}{a^3}\left(3\cdot x_0{}^2y_0 - y_0{}^3\right)$$
$$B_v = \frac{B_4}{a^3}\left[(u+x_0)^3 - 3\cdot(u+x_0)(v+y_0)^2\right]$$
$$= \frac{B_4}{a^3}\left(u^3 - uv^2\right) + \frac{3x_0B_4}{a^3}\left(u^2-v^2\right) + \frac{3y_0B_4}{a^3}(-2uv) +$$
$$\frac{3\cdot\left(x_0{}^2 - y_0{}^2\right)\cdot B_4}{a^3}u + \frac{-6x_0y_0B_4}{a^3}v + \frac{B_4}{a^3}\left(x_0{}^3 - 3x_0y_0{}^2\right)$$

Multipole strengths from octupole feed-down

	Sextuple	Quadrupole	Dipole
Normal	$\frac{3x_0B_4}{a^3}$	$\frac{3\cdot\left(x_0^2-y_0^2\right)B_4}{a^3}$	$\frac{B_4}{a^3}\left(x_0{}^3 - 3x_0y_0{}^2\right)$
Skew	$\frac{-3y_0B_4}{a^3}$	$\frac{-6x_0y_0B_4}{a^3}$	$\frac{B_4}{a^3}\left(3\cdot x_0{}^2y_0 - y_0{}^3\right)$

Skew

$$B_u = \frac{-A_4}{a^3}\left[(u+x_0)^3 - 3\cdot(u+x_0)(v+y_0)^2\right]$$
$$= \frac{-A_4}{a^3}\left(u^3 - uv^2\right) + \frac{-3x_0A_4}{a^3}\left(u^2 - v^2\right) + \frac{3y_0A_4}{a^3}(2uv) +$$
$$\frac{-3\cdot\left({x_0}^2 - {y_0}^2\right)\cdot A_4}{a^3}u + \frac{6x_0y_0A_4}{a^3}v + \frac{-A_4}{a^3}\left({x_0}^3 - 3x_0{y_0}^2\right)$$

$$B_v = \frac{A_4}{a^3}\left[3\cdot(u+x_0)^2(v+y_0) - (v+y_0)^3\right]$$
$$= \frac{A_4}{a^3}\left(3\cdot u^2v - v^3\right) + \frac{3x_0A_4}{a^3}(2uv) + \frac{3y_0A_4}{a^3}\left(u^2 - v^2\right)$$
$$+\frac{6x_0y_0A_4}{a^3}u + \frac{3\cdot\left(x_0^2 - y_0^2\right)A_4}{a^3}v + \frac{A_4}{a^3}\left(3\cdot{x_0}^2y_0 - {y_0}^3\right)$$

Multipole strengths from skew octupole feed-down

	Sextuple	Quadrupole	Dipole
Normal	$\frac{3y_0A_4}{a^3}$	$\frac{6x_0y_0A_4}{a^3}$	$\frac{A_4}{a^3}\left(3\cdot{x_0}^2y_0 - {y_0}^3\right)$
Skew	$\frac{3x_0A_4}{a^3}$	$\frac{3\cdot\left(x_0^2 - y_0^2\right)A_4}{a^3}$	$\frac{-A_4}{a^3}\left({x_0}^3 - 3x_0{y_0}^2\right)$

Decapole

Normal

$$B_u = \frac{B_5}{a^4}\left[4\cdot(u+x_0)^3\cdot(v+y_0) - 4\cdot(u+x_0)\cdot(v+y_0)^3\right]$$
$$= \frac{B_5}{a^4}\left(4u^3v - 4uv^3\right) + \frac{4x_0B_5}{a^4}\left(3u^2v - v^3\right) + \frac{4y_0B_5}{a^4}\left(u^3 - 3uv^2\right)$$
$$+\frac{6\cdot\left({x_0}^2 - {y_0}^2\right)\cdot B_5}{a^4}(2uv) + \frac{6\cdot(2x_0y_0)B_5}{a^4}\left(u^2 - v^2\right)$$
$$+\frac{4\cdot\left(3{x_0}^2y_0 - {y_0}^3\right)\cdot B_5}{a^4}u + \frac{4\cdot\left({x_0}^3 - 3x_0{y_0}^2\right)\cdot B_5}{a^4}v + \frac{B_5}{a^4}\left(4{x_0}^3y_0 - 4x_0{y_0}^3\right)$$

$$B_v = \frac{B_5}{a^4}\left[(u+x_0)^4 - 6\cdot(u+x_0)^2(v+y_0)^2 + (v+y_0)^2\right]$$
$$= \frac{B_5}{a^4}\left(u^4 - 6u^2v^2 + v^4\right) + \frac{4x_0B_5}{a^4}\left(u^3 - 3uv^2\right) + \frac{-4y_0B_5}{a^4}\left(3u^2v - v^3\right)$$
$$+\frac{6\cdot\left({x_0}^2 - {y_0}^2\right)\cdot B_5}{a^4}\left(u^2 - v^2\right) + \frac{6\cdot(2x_0y_0)B_5}{a^4}(-2uv)$$
$$+\frac{4\cdot\left({x_0}^3 - 3x_0{y_0}^2\right)\cdot B_5}{a^4}u + \frac{-4\cdot\left(3{x_0}^2y_0 - {y_0}^3\right)\cdot B_5}{a^4}v + \frac{B_5}{a^4}\left({x_0}^4 - 6{x_0}^2{y_0}^2 + {y_0}^4\right)$$

Multipole strengths from decapole feed-down

	Octupole	Sextuple	Quadrupole	Dipole
Normal	$\frac{4x_0B_5}{a^4}$	$\frac{6\cdot\left(x_0^{\ 2}-y_0^{\ 2}\right)\cdot B_5}{a^4}$	$\frac{4\cdot\left(x_0^{\ 3}-3x_0y_0^{\ 2}\right)\cdot B_5}{a^4}$	$\frac{B_5}{a^4}\left(x_0^{\ 4}-6x_0^{\ 2}y_0^{\ 2}+y_0^{\ 4}\right)$
Skew	$\frac{-4y_0B_5}{a^4}$	$\frac{-6\cdot(2x_0y_0)\cdot B_5}{a^4}$	$\frac{-4\cdot\left(3x_0^{\ 2}y_0-y_0^{\ 3}\right)\cdot B_5}{a^4}$	$\frac{B_5}{a^4}\left(4x_0^{\ 3}y_0-x_0y_0^{\ 3}\right)$

Skew

$$
\begin{aligned}
B_u &= \frac{-A_5}{a^4}\left[(u+x_0)^4-6\cdot(u+x_0)^2(v+y_0)^2+(v+y_0)^2\right] \\
&= \frac{-A_5}{a^4}\left(u^4-6u^2v^2+v^4\right)+\frac{-4x_0A_5}{a^4}\left(u^3-3uv^2\right)+\frac{4y_0A_5}{a^4}\left(3u^2v-v^3\right) \\
&\quad+\frac{-6\cdot\left(x_0^{\ 2}-y_0^{\ 2}\right)\cdot A_5}{a^4}\left(u^2-v^2\right)+\frac{6\cdot(2x_0y_0)A_5}{a^4}(2uv) \\
&\quad+\frac{-4\cdot\left(x_0^{\ 3}-3x_0y_0^{\ 2}\right)\cdot A_5}{a^4}u+\frac{4\cdot\left(3x_0^{\ 2}y_0-y_0^{\ 3}\right)\cdot A_5}{a^4}v+\frac{-A_5}{a^4}\left(x_0^{\ 4}-6x_0^{\ 2}y_0^{\ 2}+y_0^{\ 4}\right)
\end{aligned}
$$

$$
\begin{aligned}
B_v &= \frac{A_5}{a^4}\left[4\cdot(u+x_0)^3\cdot(v+y_0)-4\cdot(u+x_0)\cdot(v+y_0)^3\right] \\
&= \frac{A_5}{a^4}\left(4u^3v-4uv^3\right)+\frac{4x_0A_5}{a^4}\left(3u^2v-v^3\right)+\frac{4y_0A_5}{a^4}\left(u^3-3uv^2\right) \\
&\quad+\frac{6\cdot\left(x_0^{\ 2}-y_0^{\ 2}\right)\cdot A_5}{a^4}(2uv)+\frac{6\cdot(2x_0y_0)\cdot A_5}{a^4}\left(u^2-v^2\right) \\
&\quad+\frac{4\cdot\left(3x_0^{\ 2}y_0-y_0^{\ 3}\right)\cdot A_5}{a^4}u+\frac{4\cdot\left(x_0^{\ 3}-3x_0y_0^{\ 2}\right)\cdot A_5}{a^4}v+\frac{A_5}{a^4}\left(4x_0^{\ 2}y_0-4x_0y_0^{\ 3}\right)
\end{aligned}
$$

Multipole strengths from skew decapole feed-down

	Octupole	Sextuple	Quadrupole	Dipole
Normal	$\frac{4y_0A_5}{a^4}$	$\frac{6\cdot(2x_0y_0)\cdot A_5}{a^4}$	$\frac{4\cdot\left(3x_0^{\ 2}y_0-y_0^{\ 3}\right)\cdot A_5}{a^4}$	$\frac{A_5}{a^4}\left(4x_0^{\ 3}y_0-x_0y_0^{\ 3}\right)$
Skew	$\frac{4x_0A_5}{a^4}$	$\frac{6\cdot\left(x_0^{\ 2}-y_0^{\ 2}\right)\cdot A_5}{a^4}$	$\frac{4\cdot\left(x_0^{\ 3}-3x_0y_0^{\ 2}\right)\cdot A_5}{a^4}$	$\frac{-A_5}{a^4}\left(x_0^{\ 4}-6x_0^{\ 2}y_0^{\ 2}+y_0^{\ 4}\right)$