Problem Sheet I

Physical Cosmology
Part III Mathematical Tripos
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[1] The deceleration parameter is defined by

$$q_0 \equiv -\frac{\ddot{a}_0}{a_0 H_0^2}\,,$$

where a_0 is the scale factor today and H_0 the Hubble constant. Prove that in a matter dominated Universe ($\Lambda = 0$)

$$\Omega_{\mathrm{m},0} = 2q_0\,.$$

Hence show that

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[1 - 2q_0 + 2q_0 a_0/a\right]\,. \tag{1}$$

Show that the solutions to Eqn. (1) are as follows:

(A) $q_0 > 1/2$

$$H_0 t = q_0(2q_0 - 1)^{-3/2}\left[\theta - \sin\theta\right]\,,$$

where the development angle θ is defined by

$$1 - \cos\theta = \left(\frac{2q_0 - 1}{q_0}\right) a(t)/a_0\,.$$

(B) $q_0 = 1/2$

$$H_0 t = 2/3\left(\frac{a}{a_0}\right)^{3/2}\,.$$

(C) $0 < q_0 < 1/2$

$$H_0 t = q_0(1-2q_0)^{-3/2}\left[\sinh\psi - \psi\right] ,$$

where

$$\cosh\psi - 1 = \left(\frac{1-2q_0}{q_0}\right)\frac{a(t)}{a_0} .$$

[2] Show that in a spatially flat Universe, $\Omega_{\rm k} = 0$, $\Omega_{\rm m} + \Omega_\Lambda = 1$, the age of the Universe can be written as

$$t(z) = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}}\ln\left[\frac{1+\cos\theta}{\sin\theta}\right] ,$$

where

$$\tan\theta = \left(\frac{\Omega_{\rm m,0}}{\Omega_{\Lambda,0}}\right)^{1/2}(1+z)^{3/2} .$$

Hence show that the present age of the Universe is given by

$$t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}}\ln\left[\frac{1+\Omega_{\Lambda,0}^{1/2}}{(1-\Omega_{\Lambda,0})^{1/2}}\right] .$$

[3] In a Universe with a cosmological constant, the scale factor evolves according to

$$\dot{a}^2 = H_0^2 a^2\left(\Omega_{\rm m,0}a_0^3/a^3 + \Omega_{\Lambda,0} + \Omega_{\rm k,0}a_0^2/a^2\right) .$$

Show that, in a Universe with a positive cosmological constant, the expansion rate $\dot{a}$ reaches a minimum when the scale factor attains a value

$$a_{\rm min} = \left(\frac{\Omega_{\rm m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} a_0 .$$

Show that the expansion rate at this time is

$$\dot{a}_{\rm min}^2 = H_0^2 a_{\rm min}^2\left[3\Omega_{\Lambda,0} + \Omega_{\rm k,0}\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{\rm m,0}}\right)^{2/3}\right] .$$

Discuss the dynamics of a Universe in which the term in brackets is negative.

[4] If the present universe were radiation dominated (i.e., $p = \frac{1}{3}\rho$), show that we would expect $q_0 = \Omega_{\rm r,0}$ (and not $q_0 = \frac{1}{2}\Omega_{\rm m,0}$).

[5] Sketch the angular diameter vs. z relation (out to $z \gg 1$) for "rigid rods" in Friedman models with $\Omega_{\rm m,0} = 0.01$ and $\Omega_{\rm m,0} = 1$. At what z does the minimum angular diameter occur in the $\Omega_{\rm m,0} = 1$ case. What is the physical interpretation ?

[6] Show that the Doppler formula for change in frequency of light reduces to $z = -(\delta\nu)/\nu = v/c$ for non-relativistic speeds. Use this in conjunction with the Hubble's law to compute the frequency shift as a light beam emitted by one fundamental observer is observed by another at an infinitesimal separation δr. Show that the relative frequency shift $-(\delta\nu)/\nu$ is proportional to the change in scale factor of the Universe between emission and absorption. Integrate this equation to show that

$$\nu_{\rm obs} = \nu_{\rm em} \frac{a(t_{\rm em})}{a(t_{\rm obs})}$$

or in other words

$$1 + z(t) = \frac{\nu}{\nu_0} = \frac{a(t_0)}{a(t)} \, .$$

This redshift, caused by expansion of the Universe, is called the cosmological redshift, in contrast with the kinematical redshift that is caused by relative motion of the observer and emitter and the gravitational redshift that results from difference in gravitational potential at the positions of the emitter and observer.

In the real Universe we observe a combination of these redshifts as distant sources like galaxies and clusters of galaxies may be moving even apart from the uniform expansion. Show that the combined redshift due to these effects [kinematical and cosmological] is

$$1 + z_{\rm tot} = (1 + z_{\rm cosm})(1 + z_{\rm kin})$$

where the kinematical redshift relates to the "peculiar" velocity of the galaxy through the Doppler formula. The peculiar velocity is the difference of the total velocity and the Hubble expansion velocity at the position of the galaxy.

Assuming that the Hubble parameter is 75 km s^{-1} Mpc^{-1}, at what distance does the contribution of cosmological expansion to redshift exceed that caused by peculiar motions of galaxies ? [The typical peculiar velocity of galaxies is about ~ 500 km/s].