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Fitting emittance and σ_p/p.

$$S = \sum_i^N w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]^2$$

$$\frac{\partial S}{\partial \varepsilon} = \sum_i w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]\beta_i = 0$$

$$\frac{\partial S}{\partial \delta^2} = \sum_i w_i[\sigma_i^2 - (\beta_i\varepsilon + \eta_i^2\delta^2)]\eta_i^2 = 0$$

$$\begin{pmatrix} \sum_i w_i\sigma_i^2\beta_i \\ \sum_i w_i\sigma_i^2\eta_i^2 \end{pmatrix} = \begin{bmatrix} \sum_i w_i\beta_i^2 & \sum_i w_i\eta_i^2\beta_i \\ \sum_i w_i\eta_i^2\beta_i & \sum_i w_i\eta_i^4 \end{bmatrix} \cdot \begin{pmatrix} \varepsilon \\ \delta^2 \end{pmatrix}$$

$$S = \sum_i w_i[\sigma_i^4 - 2\varepsilon\sigma_i^2\beta_i - 2\delta^2\sigma_i^2\eta_i^2 + \varepsilon^2\beta_i^2 + \delta^4\eta_i^4 + 2\varepsilon\delta^2\beta_i\eta_i^2]$$

$$S = \sum_i w_i[\sigma_i^4 - 2\varepsilon\sigma_i^2\beta_i - 2\delta^2\sigma_i^2\eta_i^2 + \varepsilon^2\beta_i^2 + \delta^4\eta_i^4 + 2\varepsilon\delta^2\beta_i\eta_i^2]$$

$$= \sum_i w_i\sigma_i^4 - 2\varepsilon\sum_i w_i\sigma_i^2\beta_i - 2\delta^2\sum_i w_i\sigma_i^2\eta_i^2 + \varepsilon^2\sum_i w_i\beta_i^2 + \delta^4\sum_i w_i\eta_i^4 + 2\varepsilon\delta^2\sum_i w_i\beta_i\eta_i^2$$

$$\chi^2 = S/(N-2)$$

$$\begin{pmatrix} \varepsilon \\ \delta^2 \end{pmatrix} = \Re \cdot \begin{pmatrix} \sum_i w_i\sigma_i^2\beta_i \\ \sum_i w_i\sigma_i^2\eta_i^2 \end{pmatrix}$$

$$\Delta\varepsilon = \sqrt{\chi^2 \cdot \Re_{11}}$$

$$\Delta\delta^2 = \sqrt{\chi^2 \cdot \Re_{22}}$$

$$\Delta\delta = \frac{\sqrt{\chi^2 \cdot \Re_{22}}}{2\delta}$$

$$\Delta\delta = \sqrt{\chi^2 \cdot \Re_{22}} \Big/ 2\delta$$

Momentum deviation calculation using RF frequency

Phase slip factor η

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

The transition gamma γ_t is 5.401 for Booster and 21.6 for Main Injector.

δp/p calculation

$$\frac{\Delta p}{p} = -\frac{1}{\eta} \cdot \frac{\Delta f}{f}$$

Differential relations between *B*, *R*, *Δp/p*, and *f*

Variables	**Equations**
B, p, R	$\frac{\Delta p}{p} = \gamma_{tr}^2 \cdot \frac{\Delta R}{R} + \frac{\Delta B}{B}$
f, p, R	$\frac{\Delta p}{p} = \gamma^2 \cdot \frac{\Delta f}{f} + \gamma^2 \cdot \frac{\Delta R}{R}$
B, f, p	$\frac{\Delta B}{B} = \gamma_{tr}^2 \cdot \frac{\Delta f}{f} + \frac{\gamma^2 - \gamma_{tr}^2}{\gamma^2} \cdot \frac{\Delta p}{p}$
B, f, R	$\frac{\Delta B}{B} = \gamma^2 \cdot \frac{\Delta f}{f} + \left(\gamma^2 - \gamma_{tr}^2\right) \cdot \frac{\Delta R}{R}$

Lattice function calculation closed orbit BPM data

Phase advance calculation

The 1-bump orbit displacement at *i*-th BPM caused by the dipole correctors can be written as:

$$x_i^{(1)} = \frac{\sqrt{\beta_i \beta_1} \cdot k_1 \cos\left[2\pi(\psi_i - \alpha_1)\right]}{2\sin(\pi Q)}, \quad \text{with } \alpha_1 = \psi_1 + Q/2 \text{ and} \tag{1a}$$

$$x_i^{(2)} = \frac{\sqrt{\beta_i \beta_2} \cdot k_2 \cos\left[2\pi(\psi_i - \alpha_2)\right]}{2\sin(\pi Q)}, \quad \text{with } \alpha_2 = \psi_2 + Q/2 \text{ and} \tag{1b}$$

Q is the total tune of the machine. The indices 1 & 2 are used to denote the two dipole corrector used to cause 1-bump orbit, i.e. ψ_1 and ψ_2 are the phase at the corrector location. The indices *i* denote the *i*-th BPM location.

Define $d_i^{(1)} = \frac{2\sin(\pi Q)}{\sqrt{\beta_1} \cdot k_1} x_i^{(1)}$ and $d_i^{(2)} = \frac{2\sin(\pi Q)}{\sqrt{\beta_2} \cdot k_2} x_i^{(2)}$ and we get:

$$\cos(2\pi\psi_i) = \frac{1}{\sqrt{\beta_i}} \frac{d_i^{(1)} \sin(2\pi\alpha_2) - d_i^{(2)} \sin(2\pi\alpha_1)}{\sin\left[2\pi(\alpha_2 - \alpha_1)\right]} \tag{2a}$$

$$\sin(2\pi\psi_i) = \frac{1}{\sqrt{\beta_i}} \frac{d_i^{(1)} \cos(2\pi\alpha_2) - d_i^{(2)} \cos(2\pi\alpha_1)}{-\sin\left[2\pi(\alpha_2 - \alpha_1)\right]} \tag{2b}$$

From equation (2) the value of ψ_i can be calculated without knowing what β_i is.

beta calculation

Square-sum of equation (2a) and (2b) provides the equation to calculate β_i with:

$$\beta_i = \left[\frac{d_i^{(1)} \cos(2\pi\alpha_2) - d_i^{(2)} \cos(2\pi\alpha_1)}{-\sin\left[2\pi(\alpha_2 - \alpha_1)\right]}\right]^2 + \left[\frac{d_i^{(1)} \sin(2\pi\alpha_2) - d_i^{(2)} \sin(2\pi\alpha_1)}{\sin\left[2\pi(\alpha_2 - \alpha_1)\right]}\right]^2 \tag{3a}$$

Continuing on with equation (3a),

$$\beta_i = \frac{\left(d_i^{(1)}\right)^2 + \left(d_i^{(2)}\right)^2 - 2 \cdot d_i^{(1)} \cdot d_i^{(2)} \cdot \cos\left[2\pi\left(\alpha_1 - \alpha_2\right)\right]}{\sin^2\left[2\pi(\alpha_2 - \alpha_1)\right]} \tag{3b}$$

evaluating the derivatives of β_i as:

$$\frac{\partial \beta_i}{\partial\left(d_i^{(1)}\right)} = \frac{2d_i^{(1)} - 2 \cdot d_i^{(2)} \cdot \cos\left[2\pi\left(\alpha_1 - \alpha_2\right)\right]}{\sin^2\left[2\pi(\alpha_2 - \alpha_1)\right]}$$

$$\frac{\partial \beta_i}{\partial\left(d_i^{(2)}\right)} = \frac{2d_i^{(2)} - 2 \cdot d_i^{(1)} \cdot \cos\left[2\pi\left(\alpha_1 - \alpha_2\right)\right]}{\sin^2\left[2\pi(\alpha_2 - \alpha_1)\right]}$$

and the error on β_i can be written as:

$$\left(\Delta\beta_i\right)^2 = \left[\frac{\partial\beta_i}{\partial\left(d_i^{(1)}\right)}\right]^2 \cdot \left(\Delta d_i^{(1)}\right)^2 + \left[\frac{\partial\beta_i}{\partial\left(d_i^{(2)}\right)}\right]^2 \cdot \left(\Delta d_i^{(2)}\right)^2 \tag{4}$$

Ring closure calculation

Twiss parameters

Treating a ring as transfer line and start with initial twiss parameter of β_1 and α_1, we get at one turn around β_2, α_2, and phase advance ψ_{12}. The 1-turn transfer matrix is:

$$M = \begin{bmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) & \left(\beta_1\beta_2\right)^{\frac{1}{2}} \sin\psi_{12} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{12} - \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\psi_{12} & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right) \end{bmatrix} \tag{1}$$

This same matrix can be expressed in the actual ring twiss parameters β, α, and total phase advance ψ as:

$$M = \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \tag{2}$$

From equation (1) and (2) we get:

$$\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) = \cos\psi + \alpha\sin\psi \tag{3a}$$

$$\left(\beta_1\beta_2\right)^{\frac{1}{2}} \sin\psi_{12} = \beta\sin\psi \tag{3b}$$

$$-\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi_{12} - \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\psi_{12} = -\frac{1+\alpha^2}{\beta}\sin\psi \tag{3c}$$

$$\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right) = \cos\psi - \alpha\sin\psi \tag{3d}$$

Using equ. (3a) and (3d) we get:

$$\begin{aligned} \cos\psi &= 0.5 \cdot \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\left(\cos\psi_{12} + \alpha_1 \sin\psi_{12}\right) + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\left(\cos\psi_{12} - \alpha_2 \sin\psi_{12}\right)\right] \\ &= 0.5 \cdot \left\{\left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\right] \cdot \cos\psi_{12} + \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\alpha_1 - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\alpha_2\right] \cdot \sin\psi_{12}\right\} \end{aligned} \tag{4a}$$

and

$$\sin\psi = \sqrt{1-\cos^2\psi} \tag{4b}$$

With equation (3b) we have:

$$\beta = \frac{(\beta_1\beta_2)^{\frac{1}{2}} \sin\psi_{12}}{\sin\psi} \tag{5}$$

with the sign of $\sin\psi$ being such that β is positive.

By subtracting equ. (3d) from (3a) the value of α follows:

$$\alpha = \frac{\left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}} - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\right]\cdot\cos\psi_{12} + \left[\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{2}}\alpha_1 + \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\alpha_2\right]\cdot\sin\psi_{12}}{2\sin\psi} \tag{6}$$

Dispersion functions

In similar approach we treat the ring as beamline and track the dispersion function all the way around with η_1, η_1', being the initial condition and η_2, η_2' being the result at 1-turn. To find the right initial condition such that the initial and final condition are the same we can write down the equation as:

$$\begin{bmatrix}\eta_2\\ \eta_2'\end{bmatrix} + \begin{bmatrix}\cos\psi + \alpha\sin\psi & \beta\sin\psi\\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi\end{bmatrix}\begin{pmatrix}\Delta\eta\\ \Delta\eta'\end{pmatrix} = \begin{pmatrix}\eta_1 + \Delta\eta\\ \eta_1' + \Delta\eta'\end{pmatrix} = \begin{pmatrix}\eta\\ \eta'\end{pmatrix}$$

where η and η' are the closed solution. Substituting $\Delta\eta$ with $\eta - \eta_1$ and $\Delta\eta'$ with $\eta' - \eta_1'$ we get:

$$\begin{bmatrix}\eta_2\\ \eta_2'\end{bmatrix} + \begin{bmatrix}\cos\psi + \alpha\sin\psi & \beta\sin\psi\\ -\frac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi\end{bmatrix}\begin{pmatrix}\eta - \eta_1\\ \eta' - \eta_1'\end{pmatrix} = \begin{pmatrix}\eta\\ \eta'\end{pmatrix}$$

This gives us two equations from with to solve for η and η':

$$\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}\begin{pmatrix}\eta\\ \eta'\end{pmatrix} = \begin{pmatrix}a_0\\ b_0\end{pmatrix}$$

where

$$a_0 = \eta_2 - \left[(\cos\psi + \alpha\sin\psi)\cdot\eta_1 + \beta\sin\psi\cdot\eta_1'\right]$$

$$a_1 = 1 - \cos\psi - \alpha\sin\psi$$

$$a_2 = -\beta\sin\psi$$

$$b_0 = \eta_2' - \left[\frac{-(1+\alpha^2)}{\beta}\sin\psi\cdot\eta_1 + (\cos\psi - \alpha\sin\psi)\cdot\eta_1'\right]$$

$$b_1 = \frac{1+\alpha^2}{\beta}\sin\psi$$

$$b_2 = 1 - \cos\psi + \alpha\sin\psi$$

Closed orbit

The equation to solve for the closed orbit is exactly the same as that of the dispersion function, except that η is replaced by x.

$$\begin{bmatrix} x_2 \\ x_2^{'} \end{bmatrix} + \begin{bmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\dfrac{1+\alpha^2}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{bmatrix} \begin{pmatrix} x - x_1 \\ x^{'} - x_1^{'} \end{pmatrix} = \begin{pmatrix} x \\ x^{'} \end{pmatrix}$$

Among the coefficients above only a_0 and b_0 need to be recalculated. The initial condition for the horizontal plane and vertical plane closed orbit can be solved the same way.

Formula for calculating closed bumps

3-Bump for desired displacement Δx

$$\theta_1 = \frac{\Delta x}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\theta_2 = -\theta_1\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\theta_3 = \theta_1\left(\frac{\beta_1}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

4-Bump of equal displacement at location 2 & 3

Use the equations above we can construct two adjacent 3-bumps. The first is:

$$\alpha_1 = \frac{\Delta x_2}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\alpha_2 = -\alpha_1\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

$$\alpha_3 = \alpha_1\left(\frac{\beta_1}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

By making $\Delta x_3 = \Delta x_2 = \alpha_1\sqrt{\beta_1\beta_2}\,\sin\psi_{12}$ and with final 4-bump angles as: $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2 + \gamma_2$, $\theta_3 = \alpha_3 + \gamma_3$, and $\theta_4 = \alpha_4$, the second 3-bump can be written as:

$$\gamma_2 = \frac{\Delta x_3}{\sqrt{\beta_2\beta_3}\,\sin\psi_{23}} = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

$$\gamma_3 = -\gamma_2\left(\frac{\beta_2}{\beta_3}\right)^{\frac{1}{2}}\frac{\sin\psi_{24}}{\sin\psi_{34}} = -\theta_1\frac{\sqrt{\beta_1\beta_2}}{\beta_3}\,\frac{\sin\psi_{12}\cdot\sin\psi_{24}}{\sin\psi_{23}\cdot\sin\psi_{34}}$$

$$\gamma_4 = \gamma_2\left(\frac{\beta_2}{\beta_4}\right)^{\frac{1}{2}}\frac{\sin\psi_{23}}{\sin\psi_{34}} = \theta_1\sqrt{\frac{\beta_1\beta_2}{\beta_3\beta_4}}\,\frac{\sin\psi_{12}}{\sin\psi_{34}}$$

,

Angles for the 4-bump

$$\theta_1 = \alpha_1 = \frac{\Delta x_2}{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}$$

$$\theta_2 = \alpha_2 + \gamma_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\,\frac{\sin\psi_{13}}{\sin\psi_{23}} + \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}} = \frac{-\theta_1\sqrt{\beta_1}}{\sin\psi_{23}}\left(\frac{\sin\psi_{13}}{\sqrt{\beta_2}} - \frac{\sin\psi_{12}}{\sqrt{\beta_3}}\right)$$

$$\theta_3 = \alpha_3 + \gamma_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}} - \theta_1\frac{\sqrt{\beta_1\beta_2}}{\beta_3}\,\frac{\sin\psi_{12}\cdot\sin\psi_{24}}{\sin\psi_{23}\cdot\sin\psi_{34}} = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\,\frac{\sin\psi_{12}}{\sin\psi_{23}}\left(1 - \sqrt{\frac{\beta_2}{\beta_3}}\,\frac{\sin\psi_{24}}{\sin\psi_{34}}\right)$$

$$\theta_4 = \gamma_4 = \theta_1\frac{\sqrt{\beta_1\beta_2}\,\sin\psi_{12}}{\sqrt{\beta_3\beta_4}\,\sin\psi_{34}}$$

Displacements within the 4-bump at any K location

$\Delta x_K = \sum_{i=1,2,3,4}\theta_i\sqrt{\beta_i\beta_K}\,\sin\psi_{iK}$, for all i's with $\psi_{iK} > 0$.

Calculation of multi-moments, sigma, skewness and kurtosis

Discrete n-th moment calculation

Using the cumulant, or raw moments

$$s_n = \sum_i x_i^n y_i$$

and the mean

$$\bar{x} = \frac{1}{s_0} \sum_i x_i y_i = \frac{s_1}{s_0}$$

The n-th moment $\mu_n = \frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^n \cdot y_i$ can be calculated.

Sigma

$$\sigma^2 = \mu_2 = \frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^2 \cdot y_i = \frac{1}{s_0} \sum_i x_i^2 y_i - \bar{x}^2 = \frac{s_2}{s_0} - \bar{x}^2$$

skewness

$$\zeta = \frac{\mu_3}{\sigma^3} = \frac{1}{\sigma^3} \left[\frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^3 \cdot y_i \right] = \frac{1}{\sigma^3} \left[\frac{s_3}{s_0} - 3\bar{x} \cdot \frac{s_2}{s_0} + 2\bar{x}^3 \right]$$

Kurtosis

$$\kappa = \frac{\mu_4}{\sigma^4} = \frac{1}{\sigma^4} \left[\frac{1}{s_0} \sum_i \left(x_i - \bar{x}\right)^4 \cdot y_i \right] = \frac{1}{\sigma^4} \left[\frac{s_4}{s_0} - 4\bar{x} \cdot \frac{s_3}{s_0} + 6\bar{x}^2 \cdot \frac{s_2}{s_0} - 3\bar{x}^4 \right]$$

and the ***excess kurtosis***

$\kappa_{ex} = \kappa - 3.$

Rotation of ellipses and formula

Ellipse in polar coordinate

The equation for an un-rotated ellipse, i.e. major axis on the x-axis, can be written down as $r = \dfrac{a^2 - f^2}{a - f \cdot \cos\theta}$, with a being the half length of major axis and f being the distance from center of ellipse to the focal point. It is also conventionally written as $r = \dfrac{p}{1 - e \cdot \cos\theta}$.

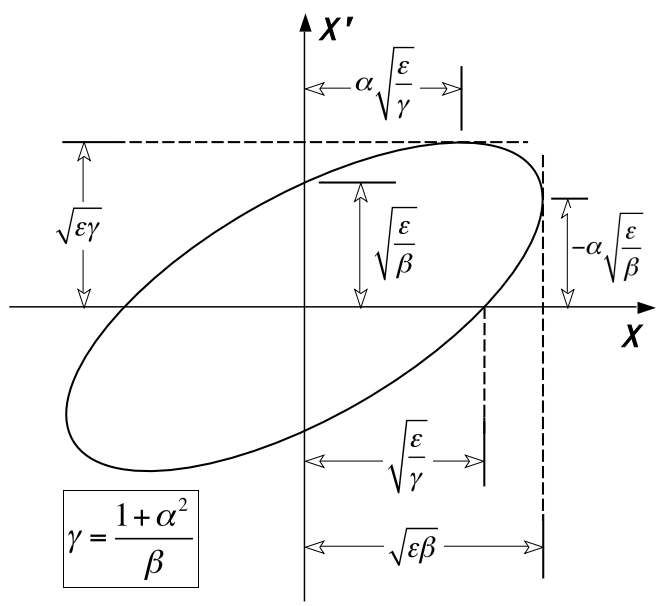


Figure 1: Characteristics of an elliptical contour.

Ellipse in parametric formula, using lattice parameters

$$\begin{cases} x = \sqrt{\varepsilon\beta}\cos\phi \\ x' = -\sqrt{\dfrac{\varepsilon}{\beta}}(\sin\phi + \alpha \cdot \cos\phi) \end{cases} \tag{1}$$

Where β and α are lattice parameters, and ε the emittance.

Rotate ellipse to zero tilt

To un-tilt a rotated ellipse the angle of the major axis need to be determined. This can be accomplished by utilizing the fact that

$$\vec{r}_A \bullet \Delta(\vec{r}_A) = 0 , \tag{2}$$

Where $\vec{r}_A = \begin{bmatrix} x_A \\ x'_A \end{bmatrix}$, using expression given in (1), is the vector from origin of ellipse to point A, where major axis intersects the ellipse. The other vector $\Delta(\vec{r}_A) = \begin{bmatrix} \frac{dx}{d\phi} \\ \frac{dx'}{d\phi} \end{bmatrix}_A$ is the derivative of $\vec{r}_A$ and is given as

$$\begin{cases} \frac{dx}{d\phi} = -\sqrt{\varepsilon\beta}\sin\phi_A \\ \frac{dx'}{d\phi} = -\sqrt{\frac{\varepsilon}{\beta}}\left(\cos\phi_A - \alpha\sin\phi_A\right) \end{cases}, \tag{3}$$

with ϕ_A being the phase at point A.
Applying expression (1) and (3) in the dot-product equation (2) leads to

$$\tan\left(2\phi_A\right) = \frac{2\alpha}{\beta^2 + \alpha^2 - 1},$$

The direction of the major axis can now be calculated as:

$$\theta = \tan^{-1}\left(\frac{x'_A}{x_A}\right) = \tan^{-1}\left(\frac{-\sqrt{\frac{\varepsilon}{\beta}}(\sin\phi_A + \alpha\cdot\cos\phi_A)}{\sqrt{\varepsilon\beta}\cos\phi_A}\right) = \tan^{-1}\left(\frac{-\left(\sin\phi_A + \alpha\cdot\cos\phi_A\right)}{\beta\cdot\cos\phi_A}\right),$$

and the half-length of major axis a as:

$$a = \sqrt{\left(\sqrt{\varepsilon\beta}\cos\phi_A\right)^2 + \left(-\sqrt{\frac{\varepsilon}{\beta}}(\sin\phi_A + \alpha\cdot\cos\phi_A)\right)^2}$$

The new beta function β_0 of the un-tilted ellipse, call it a ***normal ellipse*** for now, can be written as:

$$\beta_0 = \frac{a^2}{\varepsilon} = \beta\cos^2\phi_A + \frac{1}{\beta}\left(\sin\phi_A + \alpha\cdot\cos\phi_A\right)^2,$$

with $\alpha_0 = 0$ by default.

Rotating an ellipse

An ellipse is typically represented by:

$$\begin{cases} x = \sqrt{\varepsilon\beta}\cos\phi \\ x' = -\sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right) \end{cases}$$

Standard rotational transformation can be used to rotate the ellipse by an angle θ:

$$\begin{bmatrix} x_\theta(\phi) \\ x_\theta'(\phi) \end{bmatrix} = \begin{bmatrix} x\cos\theta + x'\sin\theta \\ -x\sin\theta + x'\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\varepsilon\beta}\cos\phi\cos\theta - \sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\sin\theta \\ -\sqrt{\varepsilon\beta}\cos\phi\sin\theta - \sqrt{\dfrac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\cos\theta \end{bmatrix}.$$

The point on the ellipse with $x_\theta = x_{Max} = \sqrt{\varepsilon\beta_\theta}$ can be found by taking the derivative of $x_\theta(\phi)$ with respect to ϕ,

$$\frac{dx_\theta(\phi)}{d\phi} = \frac{d}{d\phi}\left(\sqrt{\varepsilon\beta}\cos\phi\cos\theta + \left(-\sqrt{\frac{\varepsilon}{\beta}}\right)\left(\sin\phi + \alpha\cos\phi\right)\cdot\sin\theta\right)$$

$$= -\sqrt{\varepsilon\beta}\sin\phi\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi\sin\theta.$$

The condition that $\left[\dfrac{dx_\theta(\phi)}{d\phi}\right]_{\phi=\phi_{X\max}} = 0$ leads to:

$$\left[\frac{dx_\theta(\phi)}{d\phi}\right]_{\phi=\phi_{X\max}} = -\sqrt{\varepsilon\beta}\sin\phi_{X\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X\max}\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi_{X\max}\sin\theta$$

$$= \sin\phi_{X\max}\cdot\left(-\sqrt{\varepsilon\beta}\cos\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\theta\right) - \cos\phi_{X\max}\cdot\left(\sqrt{\frac{\varepsilon}{\beta}}\sin\theta\right)$$

$$= 0$$

and

$$\phi_{X\max} = \tan^{-1}\left(\frac{-\tan\theta}{\beta - \alpha\tan\theta}\right).$$

Using the characteristics shown in Figure 1 the new β_θ and α_θ for the rotated ellipse can be calculated:

$$x_\theta\left(\phi_{X\max}\right) = \sqrt{\varepsilon\beta}\cos\phi_{X\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X\max} + \alpha\cos\phi_{X\max}\right)\cdot\sin\theta$$

$$= \sqrt{\varepsilon\beta_\theta}$$

$$x_\theta'(\phi_{X\max}) = -\sqrt{\varepsilon\beta}\cos\phi_{X\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X\max} + \alpha\cos\phi_{X\max}\right)\cdot\cos\theta$$

$$= -\alpha_\theta \cdot \sqrt{\frac{\varepsilon}{\beta_\theta}}$$

Similarly setting the derivative of $x_\theta'(\phi)$ to zero,

$$\left.\frac{d\left[x_\theta'(\phi)\right]}{d\phi}\right|_{\phi=\phi_{X'\max}} = \left.\frac{d}{d\phi}\left(-\sqrt{\varepsilon\beta}\cos\phi\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\cdot\cos\theta\right)\right|_{\phi=\phi_{X'\max}}$$

$$= \sqrt{\varepsilon\beta}\sin\phi_{X'\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X'\max}\cos\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\sin\phi_{X'\max}\cos\theta$$

$$= \sin\phi_{X'\max}\left(\sqrt{\varepsilon\beta}\sin\theta + \alpha\sqrt{\frac{\varepsilon}{\beta}}\cos\theta\right) - \sqrt{\frac{\varepsilon}{\beta}}\cos\phi_{X'\max}\cos\theta$$

$$= 0$$

will then lead to

$$\phi_{X'\max} = \tan^{-1}\left(\frac{1}{\beta\cdot\tan\theta + \alpha}\right).$$

Plugging in $\phi_{X'\max}$ to get

$$x_\theta'(\phi_{X'\max}) = -\sqrt{\varepsilon\beta}\cos\phi_{X'\max}\sin\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X'\max} + \alpha\cos\phi_{X'\max}\right)\cdot\cos\theta$$

$$= \sqrt{\varepsilon\gamma_\theta} = \sqrt{\varepsilon\cdot\frac{1+\alpha_\theta^2}{\beta_\theta}}$$

and

$$x_\theta(\phi_{X'\max}) = \sqrt{\varepsilon\beta}\cos\phi_{X'\max}\cos\theta - \sqrt{\frac{\varepsilon}{\beta}}\left(\sin\phi_{X'\max} + \alpha\cos\phi_{X'\max}\right)\cdot\sin\theta$$

$$= \alpha_\theta\sqrt{\frac{\varepsilon}{\gamma_\theta}} = \alpha_\theta\sqrt{\frac{\varepsilon\beta_\theta}{1+\alpha_\theta^2}}.$$

new lattice function β_θ and α_θ can be calculated accordingly:

$$\alpha_\theta = \frac{x_\theta(\phi_{X'\max})\cdot x'_\theta(\phi_{X'\max})}{\varepsilon}$$

$$\gamma_\theta = \frac{\left[x'_\theta(\phi_{X'\max})\right]^2}{\varepsilon}$$

$$\beta_\theta = \frac{1+\alpha_\theta^2}{\gamma_\theta}$$

Fitting profile sigma for emittance and Δp/p

Formula

The observed beam profile sigma σ_i at the *i*-th profile monitor can be written down as

$$\sigma_i^2 = \varepsilon \cdot \beta_i + \eta_i^2 \cdot \delta^2$$

with β_i and η_i being the corresponding beta and dispersion function. The emittance and Δp/p, i.e. ε and δ, can be found by minimizing the action sum:

$$S = \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 .$$

Taking the derivative of *S* with respective to ε we get:

$$\begin{aligned}
\frac{\partial S}{\partial \varepsilon} &= \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 \\
&= \sum_i -2\beta_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right) \\
&= \sum_i -2\beta_i \sigma_i^2 + \sum_i -2\beta_i \left(-\varepsilon \cdot \beta_i\right) + \sum_i -2\beta_i \left(-\eta_i^2 \cdot \delta^2\right) = 0 \\
\Rightarrow \varepsilon \cdot \sum_i \beta_i^2 + \delta^2 \cdot \sum_i \beta_i \eta_i^2 &= \sum_i \beta_i \sigma_i^2
\end{aligned} \tag{1}$$

Taking the derivative of *S* with respect to we get:

$$\begin{aligned}
\frac{\partial S}{\partial \delta^2} &= \sum_i \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right)^2 \\
&= \sum_i -2\eta_i^2 \left(\sigma_i^2 - \varepsilon \cdot \beta_i - \eta_i^2 \cdot \delta^2\right) \\
&= \sum_i -2\eta_i^2 \sigma_i^2 + \sum_i -2\eta_i^2 \left(-\varepsilon \cdot \beta_i\right) + \sum_i -2\eta_i^2 \left(-\eta_i^2 \cdot \delta^2\right) = 0 \\
\Rightarrow \varepsilon \cdot \sum_i \eta_i^2 \beta_i + \delta^2 \cdot \sum_i \eta_i^4 &= \sum_i \eta_i^2 \sigma_i^2
\end{aligned} \tag{2}$$

Fitting for both emittance and Δp/p

From (1) and (2) a matrix equation can be constructed as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \delta^2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3}$$

where

$$a_{11} = \sum_i \beta_i^2 , \quad a_{12} = \sum_i \beta_i \eta_i^2 ,$$

$$a_{21} = \sum_i \eta_i^2 \beta_i \quad , \quad a_{22} = \sum_i \eta_i^4 \quad ,$$

and

$$V_1 = \sum_i \beta_i \sigma_i^2 \quad , \quad V_2 = \sum_i \eta_i^2 \sigma_i^2$$

The values for ε and δ can now be found by solving the 2×2 equations (3).

Fitting only for emittance

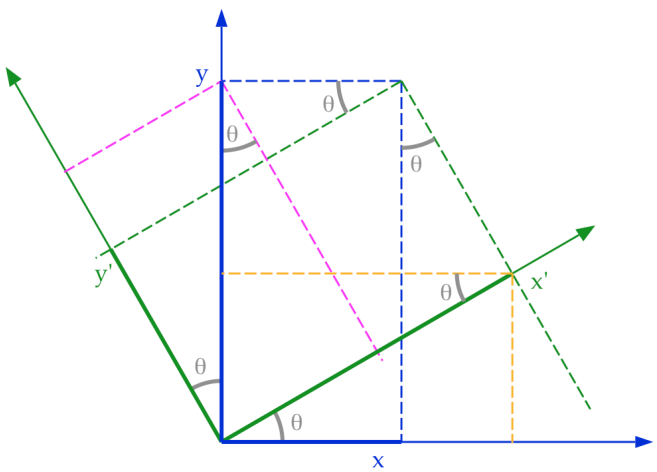
In this case δ is assumed known and ε can be calculated readily:

$$\varepsilon = \frac{V_1 - \delta^2 \cdot a_{12}}{a_{11}} \quad \text{, or}$$

$$\varepsilon = \frac{V_2 - \delta^2 \cdot a_{22}}{a_{21}}$$

Basic coordinate rotation

<u>Rotate by an angle **θ**</u>



From the drawing above we can write down the transformation matrix as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and the inverse of it:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Fitting periodic pattern with sine function

Phase and amplitude fitting

For a given oscillation frequency Ω the best matching φ, amplitude A, and overall offset δ can be determine through least square minimization.
The functional value of the oscillation can be written as:

$$\begin{aligned} f(t_i) &= A\cdot e^{-\frac{t_i}{\tau}}\cdot\sin(\Omega\cdot t_i+\varphi)+\delta \\ &= A\cdot e^{-\frac{t_i}{\tau}}\cdot\sin(\Omega\cdot t_i)\cdot\cos\varphi + A\cdot e^{-\frac{t_i}{\tau}}\cdot\cos(\Omega\cdot t_i)\cdot\sin\varphi+\delta, \\ &= a\cdot e^{-\frac{t_i}{\tau}}\cdot\sin(\Omega\cdot t_i)+b\cdot e^{-\frac{t_i}{\tau}}\cdot\cos(\Omega\cdot t_i)+\delta \\ &= a\cdot S_i+b\cdot C_i+\delta \end{aligned}$$

Where a, b, S_i, and C_i are defined as:

$$\begin{aligned} a &= A\cos\varphi \\ b &= A\sin\varphi \\ S_i &= e^{-\frac{t_i}{\tau}}\cdot\sin(\Omega\cdot t_i) \\ C_i &= e^{-\frac{t_i}{\tau}}\cdot\cos(\Omega\cdot t_i) \end{aligned}.$$

The action S, the deviation square sum, can be written as:

$$\begin{aligned} S &= \sum_i\left[y_i - f(t_i)\right]^2 \\ &= \sum_i\left[y_i - a\cdot S_i - b\cdot C_i - \delta\right]^2 \end{aligned}$$

Minimizing S with respect to the three variables gives:

$$\begin{aligned} \frac{\partial S}{\partial a} &= \sum_i -S_i\left[y_i - a\cdot S_i - b\cdot C_i - \delta\right] \\ &= -\sum_i y_i S_i + a\cdot\sum_i S_i^2 + b\cdot\sum_i S_i\cdot C_i + \delta\cdot\sum_i S_i = 0 \\ \frac{\partial S}{\partial b} &= \sum_i -C_i\left[y_i - a\cdot S_i - b\cdot C_i - \delta\right] \\ &= -\sum_i y_i C_i + a\cdot\sum_i S_i C_i + b\cdot\sum_i C_i^2 + \delta\cdot\sum_i C_i = 0 \\ \frac{\partial S}{\partial \delta} &= \sum_i -\left[y_i - a\cdot S_i - b\cdot C_i - \delta\right] \\ &= -\sum_i y_i + a\cdot\sum_i S_i + b\cdot\sum_i C_i + \delta\cdot N = 0 \end{aligned}$$

This leads to matrix equation for *a*, *b*, and δ as:

$$\mathbf{X}\cdot\begin{bmatrix} a \\ b \\ \delta \end{bmatrix} = Y$$

where

$$\mathbf{X} = \begin{bmatrix} \sum_i S_i^2 & \sum_i S_i C_i & \sum_i S_i \\ \sum_i S_i C_i & \sum_i C_i^2 & \sum_i C_i \\ \sum_i S_i & \sum_i C_i & N \end{bmatrix} \text{ and } Y = \begin{bmatrix} \sum_i y_i S_i \\ \sum_i y_i C_i \\ \sum_i y_i \end{bmatrix}.$$

The solution then becomes:

$$\begin{bmatrix} a \\ b \\ \delta \end{bmatrix} = \mathbf{X}^{-1}\cdot Y \text{, and that leads to } \begin{cases} A = \sqrt{a^2+b^2} \\ \varphi = \tan^{-1}\left(\dfrac{b}{a}\right) \end{cases}.$$

Error of fit

Action S can now be evaluated readily with the newly determined *a*, *b*, and δ:

$$\begin{aligned} S &= \sum_i \left[y_i - a\cdot S_i - b\cdot C_i - \delta\right]^2 \\ &= \sum_i y_i^2 + a^2\cdot\sum_i S_i^2 + b^2\cdot\sum_i C_i^2 + \delta^2 N \\ &\quad -2a\cdot\sum_i y_i\cdot S_i - 2b\cdot\sum_i y_i\cdot C_i - 2\delta\cdot\sum_i y_i \\ &\quad +2ab\cdot\sum_i S_i C_i + 2a\delta\cdot\sum_i S_i + 2b\delta\cdot\sum_i C_i \\ &= \left(\sum_i y_i^2\right) + a^2\cdot X_{11} + b^2\cdot X_{22} + \delta^2\cdot X_{33} \\ &\quad -2a\cdot Y_1 - 2b\cdot Y_2 - 2\delta\cdot Y_3 + 2ab\cdot X_{21} + 2a\delta\cdot X_{31} + 2b\delta\cdot X_{32} \end{aligned}$$

Scanning for the best frequency

Though best matching frequency can't be calculated through minimization one can still find its value by using iterative scanning procedure. The value of action S as a function of Ω can be used to determine the best matching frequency. A restricted frequency range should be used to avoid possible aliasing effect.

Fitting periodic pattern with two sine function

Amplitude and phase for two frequencies

When coupling is significant TBT oscillation need to be fitted with both horizontal and vertical frequencies in order for the fit RMS to be useful a meaningful gauge of goodness-of-fit.

For a given oscillation frequencies Ω_x, Ω_y, and t_τ, the best matching phase, φ_x and φ_y, amplitude A_x and A_y, and offset δ can be determine through least square minimization.

The functional value of oscillation can be written as:

$$
\begin{aligned}
f\left(t_i\right) &= A_x \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_x \cdot t_i + \varphi_x) + A_y \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_y \cdot t_i + \varphi_y) + \delta \\
&= A_x \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_x \cdot t_i) \cdot \cos\varphi_x + A_x \cdot e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega_x \cdot t_i) \cdot \sin\varphi_x \\
&\quad + A_y \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_y \cdot t_i) \cdot \cos\varphi_y + A_y \cdot e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega_y \cdot t_i) \cdot \sin\varphi_y + \delta \\
&= a_x \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_x \cdot t_i) + b_x \cdot e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega_x \cdot t_i) \\
&\quad + a_y \cdot e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega_y \cdot t_i) + b_y \cdot e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega_y \cdot t_i) + \delta \\
&= a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta
\end{aligned},
$$

with the following substitution definitions:

$$
\begin{aligned}
&a_x = A_x \cos\varphi_x, \;\; a_y = A_y \cos\varphi_y \\
&b_x = A_x \sin\varphi_x, \;\; b_y = A_y \sin\varphi_y \\
&S_{xi} = e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega \cdot t_i), \;\; S_{yi} = e^{\frac{-t_i}{\tau}} \cdot \sin(\Omega \cdot t_i) \\
&C_{xi} = e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega \cdot t_i), \;\; C_{yi} = e^{\frac{-t_i}{\tau}} \cdot \cos(\Omega \cdot t_i)
\end{aligned}.
$$

The action $\hat{S}$, the deviation square sum, can be written as:

$$
\begin{aligned}
\hat{S} &= \left[z_i - f\left(t_i\right)\right]^2 \\
&= \sum_i \left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right]^2
\end{aligned}
$$

Minimizing $\hat{S}$ with respect to the five variables gives:

$$
\begin{aligned}
\frac{\partial \hat{S}}{\partial a_x} &= \sum_i -S_{xi}\left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right] \\
&= -\sum_i z_i S_{xi} + a_x \cdot \sum_i S_{xi}^{\;2} + b_x \cdot \sum_i S_{xi} \cdot C_{xi} + a_y \cdot \sum_i S_{xi} \cdot S_{yi} + b_y \cdot \sum_i S_{xi} \cdot C_{yi} + \delta \cdot \sum_i S_{xi} = 0
\end{aligned},
$$

$$\frac{\partial \hat{S}}{\partial b_x} = \sum_i -C_{xi}\left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right]$$
$$= -\sum_i z_i C_{xi} + a_x \cdot \sum_i S_{xi} \cdot C_{xi} + b_x \cdot \sum_i C_{xi}^{\ 2} + a_y \cdot \sum_i C_{xi} \cdot S_{yi} + b_y \cdot \sum_i C_{xi} \cdot C_{yi} + \delta \cdot \sum_i C_{xi} = 0,$$

$$\frac{\partial \hat{S}}{\partial a_y} = \sum_i -S_{yi}\left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right]$$
$$= -\sum_i z_i S_{yi} + a_x \cdot \sum_i S_{xi} \cdot S_{yi} + b_x \cdot \sum_i C_{xi} \cdot S_{yi} + a_y \cdot \sum_i S_{yi}^{\ 2} + b_y \cdot \sum_i S_{yi} \cdot C_{yi} + \delta \cdot \sum_i S_{yi} = 0,$$

$$\frac{\partial \hat{S}}{\partial b_y} = \sum_i -S_{xi}\left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right]$$
$$= -\sum_i z_i C_{yi} + a_x \cdot \sum_i S_{xi} \cdot C_{yi} + b_x \cdot \sum_i C_{xi} \cdot C_{yi} + a_y \cdot \sum_i S_{yi} \cdot C_{yi} + b_y \cdot \sum_i C_{yi}^{\ 2} + \delta \cdot \sum_i C_{yi} = 0,$$

$$\frac{\partial \hat{S}}{\partial \delta} = \sum_i -\left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta\right)\right]$$
$$= -\sum_i z_i + a_x \cdot \sum_i S_{xi} + b_x \cdot \sum_i C_{xi} + a_y \cdot \sum_i S_{yi} + b_y \cdot \sum_i C_{yi} + \delta \cdot N = 0.$$

This leads to a matrix equation for *a's*, *b's*, and δ as: $Z = T \cdot \begin{bmatrix} a_x \\ b_x \\ a_y \\ b_y \\ \delta \end{bmatrix}$,

with $Z = \begin{bmatrix} \sum_i z_i S_{xi} \\ \sum_i z_i C_{xi} \\ \sum_i z_i S_{yi} \\ \sum_i z_i C_{yi} \\ \sum_i z_i \end{bmatrix}$ and $T = \begin{bmatrix} \sum_i S_{xi}^{\ 2} & \sum_i S_{xi} \cdot C_{xi} & \sum_i S_{xi} \cdot S_{yi} & \sum_i S_{xi} \cdot C_{yi} & \sum_i S_{xi} \\ \sum_i S_{xi} \cdot C_{xi} & \sum_i C_{xi}^{\ 2} & \sum_i C_{xi} \cdot S_{yi} & \sum_i C_{xi} \cdot C_{yi} & \sum_i C_{xi} \\ \sum_i S_{xi} \cdot S_{yi} & \sum_i C_{xi} \cdot S_{yi} & \sum_i S_{yi}^{\ 2} & \sum_i S_{yi} \cdot C_{yi} & \sum_i S_{yi} \\ \sum_i S_{xi} \cdot C_{yi} & \sum_i C_{xi} \cdot C_{yi} & \sum_i S_{yi} \cdot C_{yi} & \sum_i C_{yi}^{\ 2} & \sum_i C_{yi} \\ \sum_i S_{xi} & \sum_i C_{xi} & \sum_i S_{yi} & \sum_i C_{yi} & N \end{bmatrix}$.

The solution can be written as: $\begin{bmatrix} a_x \\ b_x \\ a_y \\ b_y \\ \delta \end{bmatrix} = \left[T\right]^{-1} \cdot Z$,

and from which amplitude and phase can be calculated:

$$A_x = \sqrt{a_x^2 + b_x^2}, \quad \varphi_x = \tan^{-1}\left(\frac{b_x}{a_x}\right)$$

$$A_y = \sqrt{a_y^2 + b_y^2}, \quad \varphi_y = \tan^{-1}\left(\frac{b_y}{a_y}\right)$$

Error of fit

Minimized action S can now be evaluated readily with the newly determined *a's*, *b's*, and δ:

$$\begin{aligned}
\hat{S} &= \sum_i \left[z_i - \left(a_x \cdot S_{xi} + b_x \cdot C_{xi} + a_y \cdot S_{yi} + b_y \cdot C_{yi} + \delta \right) \right]^2 \\
&= \sum_i z^2{}_i + a_x{}^2 \cdot \sum_i S_{xi}{}^2 + b_x{}^2 \cdot \sum_i C_{xi}{}^2 + a_y{}^2 \cdot \sum_i S_{yi}{}^2 + b_y{}^2 \cdot \sum_i C_{yi}{}^2 + \delta^2 N \\
&\quad - 2a_x \cdot \sum_i z_i \cdot S_{xi} - 2b_x \cdot \sum_i z_i \cdot C_{xi} - 2a_y \cdot \sum_i z_i \cdot S_{yi} - 2b_y \cdot \sum_i z_i \cdot C_{yi} - 2\delta \cdot \sum_i z_i \\
&\quad + 2a_x b_x \cdot \sum_i S_{xi} \cdot C_{xi} + 2a_x a_y \cdot \sum_i S_{xi} \cdot S_{yi} + 2a_x b_y \cdot \sum_i S_{xi} \cdot C_{yi} + 2a_x \delta \cdot \sum_i S_{xi} \\
&\quad + 2b_x a_y \cdot \sum_i C_{xi} \cdot S_{yi} + 2b_x b_y \cdot \sum_i C_{xi} \cdot C_{yi} + 2b_x \delta \cdot \sum_i C_{xi} \\
&\quad + 2a_y b_y \cdot \sum_i S_{yi} \cdot C_{yi} + 2a_y \delta \cdot \sum_i S_{yi} \\
&\quad + 2b_y \delta \cdot \sum_i C_{yi}
\end{aligned}$$

Substituting with summations already available the action $\hat{S}$ can be rewritten as:

$$\begin{aligned}
\hat{S} &= \left(\sum_i z^2{}_i \right) + a_x{}^2 \cdot T_{00} + b_x{}^2 \cdot T_{11} + a_y{}^2 \cdot T_{22} + b_y{}^2 \cdot T_{33} + \delta^2 \cdot N \\
&\quad - 2a_x \cdot Z_0 - 2b_x \cdot Z_1 - 2a_y \cdot Z_2 - 2b_y \cdot Z_3 - 2\delta \cdot Z_4 \\
&\quad + 2a_x b_x \cdot T_{01} + 2a_x a_y \cdot T_{02} + 2a_x b_y \cdot T_{03} + 2a_x \delta \cdot T_{04} \\
&\quad + 2b_x a_y \cdot T_{12} + 2b_x b_y \cdot T_{13} + 2b_x \delta \cdot T_{14} \\
&\quad + 2a_y b_y \cdot T_{23} + 2a_y \delta \cdot T_{24} \\
&\quad + 2b_y \delta \cdot T_{34}
\end{aligned}$$

Finding minimum of a 3rd order polynomial function

Motivation

The minimum of a distribution can usually be found by using a 2nd order polynomial fit to the data. However, when the distribution around the minimum is not symmetric a 3rd order fit can provide more accuracy to the location of the minimum.

Formulation

Given a 3rd order polynomial

$$y = a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$$

The condition for either a local maximum or a local minimum is:

$$y' = 3a_3 \cdot x^2 + 2a_2 \cdot x + a_1 = 0$$

This leads to:

$$x_{\pm} = \frac{-a_2 \pm \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3}$$

and

$$\begin{aligned} y_{\pm}'' &= 6a_3 \cdot x_{\pm} + 2a_2 \\ &= 6a_3 \cdot \frac{-a_2 \pm \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3} + 2a_2 \\ &= \pm 2\sqrt{a_2^2 - 3a_3 \cdot a_1} \end{aligned}$$

Since the condition for a local minimum is that $y'' > 0$ the minimum is always with:

$$x_{+} = \frac{-a_2 + \sqrt{a_2^2 - 3a_3 \cdot a_1}}{3a_3}$$

The ordering of minimum and maximum will depend on the sign of a_3, i.e. the minimum will be on the negative side of maximum if $a_3 < 0$.

Polynomial fit to two-dimensional data array

Motivation

When data is dependent on two separate variables a two-dimensional quadratic equation, or cubic equation, may be used to fit the data and obtain the minimum.

Formula for two-dimensional ***quadratic*** function

Two-parameter quadratic equation can be written as:

$$z = a_x \cdot x^2 + b_x \cdot x + a_y \cdot y^2 + b_y \cdot y + c \tag{1}$$

The action S, summing all *i*'s and all *j*'s, can be then written as:

$$S = \sum_{i,j} \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right]^2 \tag{2}$$

Taking the derivatives with respect to each parameters leads to:

$$\begin{aligned}
\frac{\partial S}{\partial a_x} &= \sum_{i,j} x_i^2 \cdot \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right] \\
&= \sum_{i,j} z_{ij} \cdot x_i^2 - a_x \cdot \sum_{i,j} x_i^4 - b_x \cdot \sum_{i,j} x_i^3 - a_y \cdot \sum_{i,j} x_i^2 \cdot y_j^2 - b_y \cdot \sum_{i,j} x_i^2 \cdot y_j - c \cdot \sum_{i,j} x_i^2 = 0 \\
\frac{\partial S}{\partial b_x} &= \sum_{i,j} x_i \cdot \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right] \\
&= \sum_{i,j} z_{ij} \cdot x_i - a_x \cdot \sum_{i,j} x_i^3 - b_x \cdot \sum_{i,j} x_i^2 - a_y \cdot \sum_{i,j} x_i \cdot y_j^2 - b_y \cdot \sum_{i,j} x_i \cdot y_j - c \cdot \sum_{i,j} x_i = 0 \\
\frac{\partial S}{\partial a_y} &= \sum_{i,j} y_i^2 \cdot \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right] \\
&= \sum_{i,j} z_{ij} \cdot y_j^2 - a_x \cdot \sum_{i,j} x_i^2 \cdot y_j^2 - b_x \cdot \sum_{i,j} x_i \cdot y_j^2 - a_y \cdot \sum_{i,j} y_j^4 - b_y \cdot \sum_{i,j} y_j^3 - c \cdot \sum_{i,j} y_j^2 = 0 \\
\frac{\partial S}{\partial b_y} &= \sum_{i,j} y_j \cdot \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right] \\
&= \sum_{i,j} z_{ij} \cdot y_j - a_x \cdot \sum_{i,j} x_i^2 \cdot y_j - b_x \cdot \sum_{i,j} x_i \cdot y_j - a_y \cdot \sum_{i,j} y_j^3 - b_y \cdot \sum_{i,j} y_j^2 - c \cdot \sum_{i,j} y_j = 0 \\
\frac{\partial S}{\partial c} &= \sum_{i,j} \left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c \right) \right] \\
&= \sum_{i,j} z_{ij} - a_x \cdot \sum_{i,j} x_i^2 - b_x \cdot \sum_{i,j} x_i - a_y \cdot \sum_{i,j} y_j^2 - b_y \cdot \sum_{i,j} y_j - c \cdot \sum_{i,j} 1 = 0
\end{aligned}$$

For M points in i-index and N points in j-index these five simultaneous equations can be written as:

$$Z = \begin{pmatrix} \sum_{i,j} z_{ij} \cdot x_i^2 \\ \sum_{i,j} z_{ij} \cdot x_i \\ \sum_{i,j} z_{ij} \cdot y_j^2 \\ \sum_{i,j} z_{ij} \cdot y_j \\ \sum_{i,j} z_{ij} \end{pmatrix} = \left[T \right] \cdot \begin{pmatrix} a_x \\ a_x \\ b_y \\ b_y \\ c \end{pmatrix}, \tag{3}$$

with T matrix elements defined as:

$$T_{00} = \sum_{i,j} x_i^4 = N \cdot \sum_i x_i^4 ,$$

$$T_{01} = T_{10} = \sum_{i,j} x_i^3 = N \cdot \sum_i x_i^3 ,$$

$$T_{02} = T_{20} = \sum_{i,j} x_i^2 \cdot y_j^2 ,$$

$$T_{03} = T_{30} = \sum_{i,j} x_i^2 \cdot y_j ,$$

$$T_{04} = T_{40} = T_{11} = \sum_{i,j} x_i^2 = N \cdot \sum_i x_i^2 ,$$

$$T_{12} = T_{21} = \sum_{i,j} x_i \cdot y_j^2 ,$$

$$T_{13} = T_{31} = \sum_{i,j} x_i \cdot y_j ,$$

$$T_{14} = T_{41} = \sum_{i,j} x_i = N \cdot \sum_i x_i ,$$

$$T_{22} = \sum_{i,j} y_j^4 = M \cdot \sum_j y_j^4 ,$$

$$T_{23} = T_{32} = \sum_{i,j} y_j^3 = M \cdot \sum_j y_j^3 ,$$

$$T_{24} = T_{42} = T_{33} = \sum_{i,j} y_j^2 = M \cdot \sum_j y_j^2 ,$$

$$T_{34} = T_{43} = \sum_{i,j} y_j = M \cdot \sum_j y_j ,$$

$$T_{44} = \sum_{i,j} 1 = M \times N .$$

Parameters of the 2-dimensional quadratic equation can then be calculated with the equation:

$$\begin{pmatrix} a_x \\ b_x \\ a_y \\ b_y \\ c \end{pmatrix} = \left[T\right]^{-1} \cdot Z \tag{4}$$

The minimized action S can now be calculated with fitted a_x, b_x, a_y, b_y, and c values:

$$\begin{aligned}
S &= \sum_{i,j}\left[z_{ij} - \left(a_x \cdot x_i^2 + b_x \cdot x_i + a_y \cdot y_j^2 + b_y \cdot y_j + c\right)\right]^2 \\
&= \sum_{i,j} z^2{}_{ij} + a_x{}^2 \cdot \sum_{i,j} x_i^4 + b_x{}^2 \cdot \sum_{i,j} x_i{}^2 + a_y{}^2 \cdot \sum_{i,j} y_j^4 + b_y{}^2 \cdot \sum_{i,j} y_j{}^2 + c^2 N \\
&\quad -2a_x \cdot \sum_{i,j} z_{ij} \cdot x_i^2 - 2b_x \cdot \sum_{i,j} z_{ij} \cdot x_i - 2a_y \cdot \sum_{i,j} z_{ij} \cdot y_j^2 - 2b_y \cdot \sum_{i,j} z_{ij} \cdot y_j - 2c \cdot \sum_{i,j} z_{ij} \\
&\quad +2a_x b_x \cdot \sum_{i,j} x_i^3 + 2a_x a_y \cdot \sum_{i,j} x_i^2 \cdot y_j^2 + 2a_x b_y \cdot \sum_{i,j} x_i^2 \cdot y_j + 2a_x c \cdot \sum_{i,j} x_i^2 \\
&\quad +2b_x a_y \cdot \sum_{i,j} x_i \cdot y_j^2 + 2b_x b_y \cdot \sum_{i,j} x_i \cdot y_j + 2b_x c \cdot \sum_{i,j} x_i \\
&\quad +2a_y b_y \cdot \sum_{i,j} y_j^3 + 2a_y c \cdot \sum_{i,j} y_j^2 \\
&\quad +2b_y c \cdot \sum_{i,j} y_j
\end{aligned}$$

Substituting with summations already available the action S can be rewritten as:

$$\begin{aligned}
S &= \left(\sum_i z^2{}_i\right) + a_x{}^2 \cdot T_{00} + b_x{}^2 \cdot T_{11} + a_y{}^2 \cdot T_{22} + b_y{}^2 \cdot T_{33} + c^2 \cdot N \\
&\quad -2a_x \cdot Z_0 - 2b_x \cdot Z_1 - 2a_y \cdot Z_2 - 2b_y \cdot Z_3 - 2c \cdot Z_4 \\
&\quad +2a_x b_x \cdot T_{01} + 2a_x a_y \cdot T_{02} + 2a_x b_y \cdot T_{03} + 2a_x c \cdot T_{04} \\
&\quad +2b_x a_y \cdot T_{12} + 2b_x b_y \cdot T_{13} + 2b_x c \cdot T_{14} \\
&\quad +2a_y b_y \cdot T_{23} + 2a_y c \cdot T_{24} \\
&\quad +2b_y c \cdot T_{34}
\end{aligned}$$

Formula for two-dimensional ***cubic*** function

Two-dimensional cubic equation can be written as:

$$z = a_x \cdot x^3 + b_x \cdot x^2 + c_x x + a_y \cdot y^3 + b_y \cdot y^2 + c_y \cdot y + d \tag{5}$$

The action S, summing all i's and all j's, can be then written as:

$$S=\sum_{i,j}\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]^2 \tag{6}$$

Taking the derivatives with respect to each parameters leads to:

$$\frac{\partial S}{\partial a_x}=\sum_{i,j}x_i^3\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot x_i^3-a_x\cdot\sum_{i,j}x_i^6-b_x\cdot\sum_{i,j}x_i^5-c_x\cdot\sum_{i,j}x_i^4-a_y\cdot\sum_{i,j}x_i^3\cdot y_j^3-b_y\cdot\sum_{i,j}x_i^3\cdot y_j^2-c_y\cdot\sum_{i,j}x_i^3\cdot y_j-d\cdot\sum_{i,j}x_i^3=0$$

$$\frac{\partial S}{\partial b_x}=\sum_{i,j}x_i^2\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot x_i^2-a_x\cdot\sum_{i,j}x_i^5-b_x\cdot\sum_{i,j}x_i^4-c_x\cdot\sum_{i,j}x_i^3-a_y\cdot\sum_{i,j}x_i^2\cdot y_j^3-b_y\cdot\sum_{i,j}x_i^2\cdot y_j^2-c_y\cdot\sum_{i,j}x_i^2\cdot y_j-d\cdot\sum_{i,j}x_i^2=0$$

$$\frac{\partial S}{\partial c_x}=\sum_{i,j}x_i\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot x_i-a_x\cdot\sum_{i,j}x_i^4-b_x\cdot\sum_{i,j}x_i^3-c_x\cdot\sum_{i,j}x_i^2-a_y\cdot\sum_{i,j}x_i\cdot y_j^3-b_y\cdot\sum_{i,j}x_i\cdot y_j^2-c_y\cdot\sum_{i,j}x_i\cdot y_j-d\cdot\sum_{i,j}x_i=0$$

$$\frac{\partial S}{\partial a_y}=\sum_{i,j}y_j^3\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot y_j^3-a_x\cdot\sum_{i,j}x_i^3\cdot y_j^3-b_x\cdot\sum_{i,j}x_i^2\cdot y_j^3-c_x\cdot\sum_{i,j}x_i\cdot y_j^3-a_y\cdot\sum_{i,j}y_j^6-b_y\cdot\sum_{i,j}y_j^5-c_y\cdot\sum_{i,j}y_j^4-d\cdot\sum_{i,j}y_j^3=0$$

$$\frac{\partial S}{\partial b_y}=\sum_{i,j}y_j^2\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot y_j^2-a_x\cdot\sum_{i,j}x_i^3\cdot y_j^2-b_x\cdot\sum_{i,j}x_i^2\cdot y_j^2-c_x\cdot\sum_{i,j}x_i\cdot y_j^2-a_y\cdot\sum_{i,j}y_j^5-b_y\cdot\sum_{i,j}y_j^4-c_y\cdot\sum_{i,j}y_j^3-d\cdot\sum_{i,j}y_j^2=0$$

$$\frac{\partial S}{\partial c_y}=\sum_{i,j}y_j\cdot\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}\cdot y_j-a_x\cdot\sum_{i,j}x_i^3\cdot y_j-b_x\cdot\sum_{i,j}x_i^2\cdot y_j-c_x\cdot\sum_{i,j}x_i\cdot y_j-a_y\cdot\sum_{i,j}y_j^4-b_y\cdot\sum_{i,j}y_j^3-c_y\cdot\sum_{i,j}y_j^2-d\cdot\sum_{i,j}y_j=0$$

$$\frac{\partial S}{\partial d}=\sum_{i,j}\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x x_i+a_y\cdot y_j^3+b_y\cdot y_j^2+c_y\cdot y_j+d\right)\right]$$
$$=\sum_{i,j}z_{ij}-a_x\cdot\sum_{i,j}x_i^3-b_x\cdot\sum_{i,j}x_i^2-c_x\cdot\sum_{i,j}x_i-a_y\cdot\sum_{i,j}y_j^3-b_y\cdot\sum_{i,j}y_j^2-c_y\cdot\sum_{i,j}y_j-d\cdot\sum_{i,j}1=0$$

For M points in i-index and N points in j-index these seven simultaneous equations can be written as:

$$Z = \begin{pmatrix} \sum_{i,j} z_{ij} \cdot x_i^3 \\ \sum_{i,j} z_{ij} \cdot x_i^2 \\ \sum_{i,j} z_{ij} \cdot x_i \\ \sum_{i,j} z_{ij} \cdot y_j^3 \\ \sum_{i,j} z_{ij} \cdot y_j^2 \\ \sum_{i,j} z_{ij} \cdot y_j \\ \sum_{i,j} z_{ij} \end{pmatrix} = \left[T \right] \cdot \begin{pmatrix} a_x \\ b_x \\ c_x \\ a_y \\ b_y \\ c_y \\ d \end{pmatrix}, \tag{7}$$

with T matrix elements defined as:

$$T_{00} = \sum_{i,j} x_i^6 ,$$

$$T_{01} = T_{10} = \sum_{i,j} x_i^5 ,$$

$$T_{02} = T_{20} = T_{11} = \sum_{i,j} x_i^4 ,$$

$$T_{03} = T_{30} = \sum_{i,j} x_i^3 \cdot y_j^3 ,$$

$$T_{04} = T_{40} = \sum_{i,j} x_i^3 \cdot y_j^2 ,$$

$$T_{05} = T_{50} = \sum_{i,j} x_i^3 \cdot y_j ,$$

$$T_{06} = T_{60} = T_{12} = T_{21} = \sum_{i,j} x_i^3 ,$$

$$T_{13} = T_{31} = \sum_{i,j} x_i^2 \cdot y_j^3 ,$$

$$T_{14} = T_{41} = \sum_{i,j} x_i^2 \cdot y_j^2 ,$$

$$T_{15} = T_{51} = \sum_{i,j} x_i^2 \cdot y_j ,$$

$$T_{16} = T_{61} = T_{22} = \sum_{i,j} x_i^2 ,$$

$$T_{23} = T_{32} = \sum_{i,j} x_i \cdot y_j^3 ,$$

$$T_{24} = T_{42} = \sum_{i,j} x_i \cdot y_j^2 ,$$

$$T_{25} = T_{52} = \sum_{i,j} x_i \cdot y_j ,$$

$$T_{26} = T_{62} = \sum_{i,j} x_i ,$$

$$T_{34} = T_{43} = \sum_{i,j} y_j^5 ,$$

$$T_{34} = T_{43} = \sum_{i,j} y_j^5 ,$$

$$T_{35} = T_{53} = T_{44} = \sum_{i,j} y_j^4 ,$$

$$T_{36} = T_{63} = T_{45} = T_{54} = \sum_{i,j} y_j^3 ,$$

$$T_{46} = T_{64} = T_{55} = \sum_{i,j} y_j^2 ,$$

$$T_{56} = T_{65} = \sum_{i,j} y_j ,$$

$$T_{66} = \sum_{i,j} 1 = M \times N .$$

Parameters of the two-dimensional cubic equation can then be calculated with the equation:

$$\begin{pmatrix} a_x \\ b_x \\ c_x \\ a_y \\ b_y \\ c_y \\ d \end{pmatrix} = \left[T \right]^{-1} \cdot Z \tag{8}$$

The minimized action S can now be calculated with fitted a_x, b_x, a_y, b_y, and c values:

$$
\begin{aligned}
S &= \sum_{i,j}\left[z_{ij}-\left(a_x\cdot x_i^3+b_x\cdot x_i^2+c_x\cdot x_i+a_y\cdot y_j^3+b_y\cdot y_i^2+c_y\cdot y_j+d\right)\right]^2 \\
&= \sum_{i,j} z^2{}_{ij}+a_x{}^2\cdot\sum_{i,j}x_i^6+b_x{}^2\cdot\sum_{i,j}x_i{}^4+c_x{}^2\cdot\sum_{i,j}x_i{}^2+a_y{}^2\cdot\sum_{i,j}y_j^6+b_y{}^2\cdot\sum_{i,j}y_j{}^4+c_y{}^2\cdot\sum_{i,j}y_j{}^2+d^2N \\
&\quad -2a_x\cdot\sum_{i,j}z_{ij}\cdot x_i^3-2b_x\cdot\sum_{i,j}z_{ij}\cdot x_i^2-2c_x\cdot\sum_{i,j}z_{ij}\cdot x_i-2a_y\cdot\sum_{i,j}z_{ij}\cdot y_j^3-2b_y\cdot\sum_{i,j}z_{ij}\cdot y_j^2-2c_y\cdot\sum_{i,j}z_{ij}\cdot y_j-2d\cdot\sum_{i,j}z_{ij} \\
&\quad +2a_xb_x\cdot\sum_{i,j}x_i^5+2a_xc_x\cdot\sum_{i,j}x_i^4+2a_xa_y\cdot\sum_{i,j}x_i^3\cdot y_j^3+2a_xb_y\cdot\sum_{i,j}x_i^3\cdot y_j^2+2a_xc_y\cdot\sum_{i,j}x_i^3\cdot y_j+2a_xd\cdot\sum_{i,j}x_i^3 \\
&\quad +2b_xc_x\cdot\sum_{i,j}x_i^3+2b_xa_y\cdot\sum_{i,j}x_i^2\cdot y_j^3+2b_xb_y\cdot\sum_{i,j}x_i^2\cdot y_j^2+2b_xc_y\cdot\sum_{i,j}x_i^2\cdot y_j+2b_xd\cdot\sum_{i,j}x_i^2 \\
&\quad +2c_xa_y\cdot\sum_{i,j}x_i\cdot y_j^3+2c_xb_y\cdot\sum_{i,j}x_i\cdot y_j^2+2c_xc_y\cdot\sum_{i,j}x_i\cdot y_j+2c_xd\cdot\sum_{i,j}x_i \\
&\quad +2a_yb_y\cdot\sum_{i,j}y_j^5+2a_yc_y\cdot\sum_{i,j}y_j^4+2a_yd\cdot\sum_{i,j}y_j^3 \\
&\quad +2b_yc_y\cdot\sum_{i,j}y_j^3+2b_yd\cdot\sum_{i,j}y_j^2 \\
&\quad +2c_yd\cdot\sum_{i,j}y_j
\end{aligned}
$$

Substituting with summations already available the action S can be rewritten as:

$$
\begin{aligned}
S &= \left(\sum_i z^2{}_i\right)+a_x{}^2\cdot T_{00}+b_x{}^2\cdot T_{11}+c_x{}^2\cdot T_{22}+a_y{}^2\cdot T_{33}+b_y{}^2\cdot T_{44}+c_y{}^2\cdot T_{55}+d^2\cdot N \\
&\quad -2a_x\cdot Z_0-2b_x\cdot Z_1-2c_x\cdot Z_2-2a_y\cdot Z_3-2b_y\cdot Z_4-2c_y\cdot Z_5-2d\cdot Z_6 \\
&\quad +2a_xb_x\cdot T_{01}+2a_xc_x\cdot T_{02}+2a_xa_y\cdot T_{03}+2a_xb_y\cdot T_{04}+2a_xc_y\cdot T_{05}+2a_xd\cdot T_{06} \\
&\quad +2b_xc_x\cdot T_{12}+2b_xa_y\cdot T_{13}+2b_xb_y\cdot T_{14}+2b_xc_y\cdot T_{15}+2b_xd\cdot T_{16} \\
&\quad +2c_xa_y\cdot T_{23}+2c_xb_y\cdot T_{24}+2c_xc_y\cdot T_{25}+2c_xd\cdot T_{26} \\
&\quad +2a_yb_y\cdot T_{34}+2a_yc_y\cdot T_{35}+2a_yd\cdot T_{36} \\
&\quad +2b_yc_y\cdot T_{45}+2b_yd\cdot T_{46} \\
&\quad +2c_yd\cdot T_{56}
\end{aligned}
$$

Calculating exact quadratic equation from 3 points

Application

For interpolating or extrapolating using existing data points.

Formulation

With

$$y_i = ax_i^2 + bx_i + c \text{, where } i = 1, 2, \text{ or } 3,$$

we can write down three difference equations:

$$y_2 - y_1 = a \cdot \left(x_2^2 - x_1^2\right) + b \cdot \left(x_2 - x_1\right)$$

$$y_3 - y_2 = a \cdot \left(x_3^2 - x_2^2\right) + b \cdot \left(x_3 - x_2\right)$$

$$y_3 - y_1 = a \cdot \left(x_3^2 - x_1^2\right) + b \cdot \left(x_3 - x_1\right),$$

and will need only two of them.

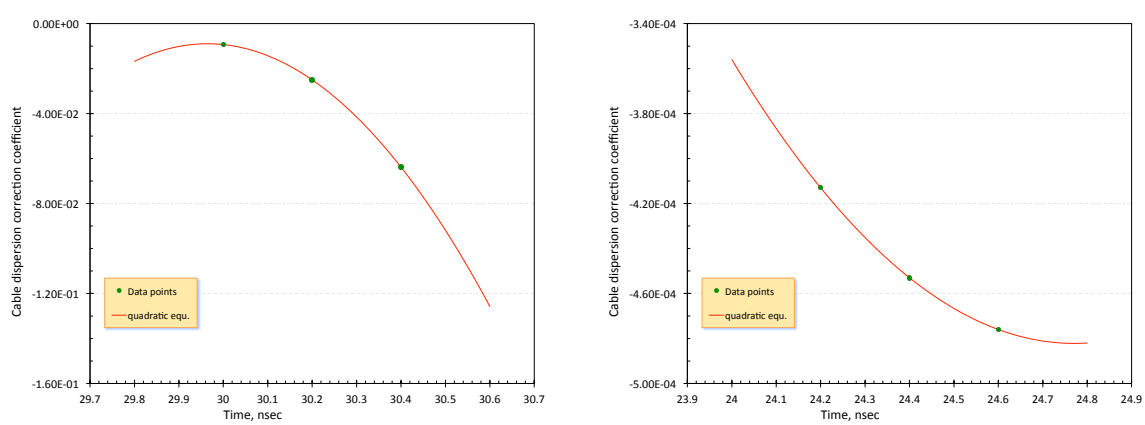
Using first two equations the solution for coefficients ***a*** and ***b*** can be written as:

$$a = \frac{\left(y_2 - y_1\right)\left(x_3 - x_2\right) - \left(y_3 - y_2\right)\left(x_2 - x_1\right)}{\left(x_2^2 - x_1^2\right)\left(x_3 - x_2\right) - \left(x_3^2 - x_2^2\right)\left(x_2 - x_1\right)}$$

and

$$b = -\frac{\left(y_2 - y_1\right)\left(x_3^2 - x_2^2\right) - \left(y_3 - y_2\right)\left(x_2^2 - x_1^2\right)}{\left(x_2^2 - x_1^2\right)\left(x_3 - x_2\right) - \left(x_3^2 - x_2^2\right)\left(x_2 - x_1\right)}.$$

The third constant ***c*** can be calculated when values for ***a*** and ***b*** have been determined.



Two data examples with their corresponding quadratic equations as calculated.