Kalman filter with a 2-D measurement

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Abstract

This note summarizes the algebra of doing a kalman filter with a pixel detector.

1 Introduction

In this example we are considering the repeating step of the kalman filter. We assume that we already have an estimator, extrapolated from previous hits, of the track parameters and the covariance matrix in the neighbourhood of the current hit. In this example, the new measurement to be added on has more than one dimension; an example is a pixel detector, for which the measurement vector has a dimension of 2. The covariance matrix of these two measurements will, in general not be diagonal. Two things which can lead to off-diagonal elements are imperfect knowledge of alignment and fluctuations in the ionization; there may be other reasons too. The notation which will be used is:

- η The track parameters, which come from previous data points projected to the current data point.
- V The covariance matrix of η.
- η' The new optimal estimator of the track parameters, after the current information has been added.
- V' The covariance matrix of η'.
- d_m The 2-d measurement.
- V_m The covariance matrix of V_m.
- $d(\eta)$ The value of the measurements, as predicted by the extrapolated track.
- $\Delta\eta$ $\eta' - \eta$
- i, j, k track parameter indices, ie η_i
- r, s measurement indices, ie d_r
- D The derivative matrix, $D_{ir} = \partial d_r / \partial \eta_i$

2 The solution

The new track parameters are found by minimizing the following χ^2 with respect to η',

$$\chi^2 \quad = \quad (d(\eta') - d_m)^T \, V_m^{-1} \, (d(\eta') - d_m) + (\Delta\eta)^T \, V^{-1} \, (\Delta\eta) \, . \tag{1}$$

The solution is found by linearizing η' about η; that is,

$$\begin{aligned} d(\eta') &= d(\eta) + D^T \Delta\eta \\ d(\eta')_r &= d(\eta)_r + (D^T)_{rj} \Delta\eta_j \end{aligned} \tag{2}$$

The next step is to set the derivatives of χ^2 to zero and to solve for $\Delta\eta$.

$$\begin{aligned} \frac{1}{2}\frac{\partial\chi^2}{\partial\eta'_k} &= D_{ks}(V_m^{-1})_{sr}\left[d_r(\eta) + (D^T)_{rj}\Delta\eta_j - (d_m)_r\right] + V_{kj}^{-1}\Delta\eta_j \\ &= 0 \end{aligned} \tag{3}$$

Now, regroup and give the solution both in component and matrix form,

$$\left(D_{ks}(V_m^{-1})_{sr}(D^T)_{rj} + V_{kj}^{-1}\right)\Delta\eta_j \quad = \quad -D_{ks}(V_m^{-1})_{sr}(d_r(\eta) - (d_m)_r) \tag{4}$$

$$\left(DV_m^{-1}D^T + V^{-1}\right)\Delta\eta \quad = \quad -D \; V_m^{-1} \, (d(\eta) - d_m) \tag{5}$$

And the solution is,

$$\begin{aligned} \Delta\eta &= -\left(DV_m^{-1}D^T + V^{-1}\right)^{-1} D \; V_m^{-1} \, (d(\eta) - d_m) \\ &= -\left[V - VD\left(V_m + D^TVD\right)^{-1} D^TV\right] D \; V_m^{-1} \, (d(\eta) - d_m) \end{aligned} \tag{6}$$

In this last step we used an identity discussed in the appendix.

Also, we recognize that the covariance matrix of η' is simply the inverse of its weight matrix,

$$V' \quad = \quad \left(DV_m^{-1}D^T + V^{-1}\right)^{-1} . \tag{7}$$

Here the first term in () is the information added to the fit by the new measurement.

Finally, we recognize the total error on the measured quantities, using both the measurements at this point and the information extrapolated from previous points,

$$V_m(total) \quad = \quad \left(V_m + D^TVD\right) \tag{8}$$

In the limit that there is only one measured quantity, then $V_m = \sigma^2$ and D becomes a column vector. In this case the equations reduce to the familiar form,

$$\eta' \quad = \quad \eta + \left[V - \frac{VDD^TV}{\sigma^2 + D^TVD}\right] D \frac{d_m - d(\eta)}{\sigma^2} \tag{9}$$

$$V' \quad = \quad V - \frac{VDD^TV}{\sigma^2 + D^TVD} \tag{10}$$

Note that I have absorb a minus sign into the sign of the residual.

A Proof of the identity

The identity to be proven is,

$$\left(DV_m^{-1}D^T + V^{-1}\right)^{-1} \;\; = \;\; V - VD\left(V_m + D^TVD\right)^{-1}D^TV. \tag{11}$$

To prove it, first show by explicit calculation that it is the right inverse.

$$\begin{aligned}&\left(DV_m^{-1}D^T + V^{-1}\right)\left(V - VD\left(V_m + D^TVD\right)^{-1}D^TV\right) = \\ &\qquad 1 + D\left[V_m^{-1} - V_m^{-1}D^TVD\left(V_m + D^TVD\right)^{-1} - \left(V_m + D^TVD\right)^{-1}\right]D^TV\end{aligned} \tag{12}$$

Now, multiply the first term in [] on its right by $(V_m + D^TVD)(V_m + D^TVD)^{-1}$ and multiply the last term in [] on its left by $V_m^{-1}V_m$. Now, regroup to obtain,

$$1 + DV_m^{-1}\left[\left(V_m + D^TVD\right) - D^TVD - V_m\right]\left(V_m + D^TVD\right)^{-1}D^TV \;\; = \;\; 1, \tag{13}$$

which proves the identity. A similar development shows that the identity is also the left inverse, which completes the proof.

B When One Measurement is a Track Segment.

Consider the case that the measurement to be added is another track segment. The notation will be that one wishes to merge (η_1, V_1) and (η_2, V_2) to obtain (η_3, V_3). This can be done as a special case of equation 5 with track 1 coresponding to the "measurment" and track 2 corresponding to the previously existing track. Therefore, $D = 1$, $V_m = V_1$, $d_m = \eta_1$, $\Delta\eta = \eta_3 - \eta_2$, $V = V_2$ and $d(\eta) = \eta_2$. Now equation 5 reads,

$$\left(V_1^{-1} + V_2^{-1}\right)(\eta_3 - \eta_2) \;\; = \;\; -V_1^{-1}(\eta_2 - \eta_1). \tag{14}$$

Now recognize, $V_3^{-1} = \left(V_1^{-1} + V_2^{-1}\right)$. Therefore,

$$\begin{aligned}\eta_3 &= \eta_2 - V_3V_1^{-1}\eta_2 + V_3V_1^{-1}\eta_1 \\ &= \eta_2 - (1 - V_3V_2^{-1})\eta_2 + V_3V_1^{-1}\eta_1 \\ &= V_3(V_1^{-1}\eta_1 + V_2^{-1}\eta_2),\end{aligned} \tag{15}$$

which is the expected form, analagous to the formula for the weighted mean of two numbers,

$$\bar{x} \;\; = \;\; \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\left(\frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right). \tag{16}$$