TRF Notes - ThinZPlaneMS

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Abstract

Notes on ThinZPlaneMS.java. There are some bugs.

1 Scattering in a Plane $\perp$ to the z Axis.

We are interested in scattering a track with momentum $\vec{p}$ with a plane perpendicular to the z-axis at a fixed value of $z = z_p$. The normal to the plane is given by $\hat{n} = (0, 0, 1)$.

The unit vector along the track direction is, $\hat{p}$, with direction cosines: (c_x, c_y, c_z), with the usual normalization, $\sqrt{c_x^2 + c_y^2 + c_z^2} = 1$.

In TRF, the track parameters used relative to a Z-plane are $(x, y, r_x, r_y, q/p)$, where (x, y) are the global coordinates of the intersection of the track with a plane at fixed z, (r_x, r_y) are the slopes of the track at this point and q/p is the charge divided by the momentum. The slopes are given in terms of the direction cosines by $r_x = c_x/c_z$ and $r_y = c_y/c_z$.

The inverse transformation is given by:

$$c_z = \frac{1}{\sqrt{1 + r_x^2 + r_y^2}} \tag{1}$$

$$c_x = r_x c_z \tag{2}$$

$$c_y = r_y c_z \tag{3}$$

The sign of c_z, when it is important, needs to be supplied by external knowledge.

To simulate multiple scattering we start by defining a new basis:

$$\hat{u} = \hat{p} \tag{4}$$

$$\hat{v} = \frac{\hat{u} \times \hat{n}}{|\hat{u} \times \hat{n}|} \tag{5}$$

$$\hat{w} = \frac{\hat{u} \times \hat{v}}{|\hat{u} \times \hat{v}|} \tag{6}$$

where $\hat{u}$ is along the track direction, $\hat{v}$ is in the plane of the scattering surface and is normal to the track direction, and where $\hat{w}$ completes a right handed orthonormal basis. In this basis multiple scattering is described by two independent scatters in two planes normal to each other; the two planes are the $(\hat{u}\hat{v})$ plane and the $(\hat{u}\hat{w})$ plane. This construction will fail when the track is perpendicular to z, in which case a different basis can be used.

For a plane perpendicular to the z axis, $\hat{n} = (0, 0, 1)$, this reduces to

$$\hat{u} = (c_x, c_y, c_z) \tag{7}$$

$$\hat{v} = \frac{(-c_y, c_x, 0)}{c_T} \tag{8}$$

$$\hat{w} = \frac{(-c_z c_x, -c_z c_y, c_T^2)}{|(-c_z c_x, -c_z c_y, c_T^2)|} \tag{9}$$

$$= \frac{(-c_z c_x, -c_z c_y, c_T^2)}{c_T} \tag{10}$$

where $c_T = \sqrt{c_x^2 + c_y^2}$.

We will now make another choice of conventions: in this new basis the vector $(0, 0, 1)$ is along the track direction. With this choice, the rotation that takes a vector in the (v, w, u) basis to the lab frame ((x, y, z) basis) is represented by a matrix whose columns are,

$$\mathbf{R} = (\hat{v}, \hat{w}, \hat{u}). \tag{11}$$

The ordering of the columns is important: $\hat{u}$ is forced into the third column by construction and the ordering of the other two columns is given by the choice of a right-handed basis.

In the (v,w,u) basis frame we define the scattered unit vector by,

$$\hat{\delta} = \frac{(\epsilon_1, \epsilon_2, 1)}{\sqrt{1 + \epsilon_1^2 + \theta_2^2}} \tag{12}$$

where ϵ_i are independent random variables chosen from a unit gaussian with $\sigma = \theta_0$. Here θ_0 is the expected RMS scattering angle, projected into a plane, as given by the PDG. This is not quite correct but is good enough for small angle scatters. We can revisit this later if necesssary.

Therefore the scattered unit vector in the lab frame is,

$$\hat{p}_l' = \delta_x \hat{v} + \delta_y \hat{w} + \delta_z \hat{u} \tag{13}$$

So the output track parameters, r_x' and r_y', are given by,

$$r_x' = \frac{\delta_x v_x + \delta_y w_x + \delta_z u_x}{\delta_x v_z + \delta_y w_z + \delta_z u_z} \tag{14}$$

$$r_y' = \frac{\delta_x v_y + \delta_y w_y + \delta_z u_y}{\delta_x v_z + \delta_y w_z + \delta_z u_z} \tag{15}$$

In these ratios, the normalization of the scattered vector cancels out and the ratios simplify to:

$$r'_x = \frac{\epsilon_1 v_x + \epsilon_2 w_x + u_x}{\epsilon_1 v_z + \epsilon_2 w_z + u_z} \tag{16}$$

$$r'_y = \frac{\epsilon_1 v_y + \epsilon_2 w_y + u_y}{\epsilon_1 v_z + \epsilon_2 w_z + u_z} \tag{17}$$

The derivatives of the new track parameters wrt the scattering angles are:

$$d_{x1} = \frac{\partial r'_x}{\partial \epsilon_1} = \frac{v_x}{\epsilon_1 v_z + \epsilon_2 w_z + u_z} - \frac{r'_x v_z}{\epsilon_1 v_z + \epsilon_2 w_z + u_z} \tag{18}$$

$$= \frac{v_x - r'_x v_z}{D} \tag{19}$$

where we have defined,

$$D = \epsilon_1 v_z + \epsilon_2 w_z + u_z. \tag{20}$$

Continuing with the derivatives gives,

$$d_{x2} = \frac{\partial r'_x}{\partial \epsilon_2} = \frac{w_x - r'_x w_z}{D} \tag{21}$$

$$d_{y1} = \frac{\partial r'_y}{\partial \epsilon_1} = \frac{v_y - r'_x v_z}{D} \tag{22}$$

$$d_{y2} = \frac{\partial r'_y}{\partial \epsilon_2} = \frac{w_y - r'_y w_z}{D} \tag{23}$$

$$\tag{24}$$

Putting in the explicit representations for planes normal to the z axis reduces the derivatives to,

$$d_{x1} = -\frac{c_y}{D c_T} \tag{25}$$

$$d_{x2} = \frac{-c_z c_x - r'_x c_T^2}{D c_T} \tag{26}$$

$$d_{y1} = \frac{c_x}{D c_T} \tag{27}$$

$$d_{y2} = \frac{-c_z c_y - r'_y c_T^2}{D c_T} \tag{28}$$

These derivatives should be evaluated for $\epsilon_1 = \epsilon_2 = 0$, which means that they should be evaluated for $r'_x = r_x$, $r'_y = r_y$ and $D = c_z$. With this, the derivatives simplify to,

$$d_{x1} = -\frac{c_y}{c_z c_T} \tag{29}$$

$$d_{x2} = \frac{-c_z c_x - r_x c_T^2}{c_z c_T} \tag{30}$$

$$= -r_x \frac{c_z^2 + c_T^2}{c_z c_T} \tag{31}$$

$$= -\frac{r_x}{c_z c_T} \tag{32}$$

$$d_{y1} = \frac{c_x}{c_z c_T} \tag{33}$$

$$d_{y2} = \frac{-c_z c_y - r_y c_T^2}{c_z c_T} \tag{34}$$

$$= -\frac{r_y}{c_z c_T} \tag{35}$$

$$\tag{36}$$

In the next step, to keep the notation clear for checking results, we will define the expectation values $< \epsilon_i^2 >= \theta_i^2$, even though we know that both $\theta_i = \theta_0$. The symmetric covariance matrix of r'_x and r'_y is given by the standard propagation formulae,

$$V = \begin{pmatrix} V_{r'_x r'_x} & V_{r'_x r'_y} \\ V_{r'_x r'_y} & V_{r'_y r'_y} \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} d_{x1} & d_{y1} \\ d_{x2} & d_{y2} \end{pmatrix}^T \begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix} \begin{pmatrix} d_{x1} & d_{y1} \\ d_{x2} & d_{y2} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} d_{x1} & d_{x2} \\ d_{y1} & d_{y2} \end{pmatrix} \begin{pmatrix} \theta_1^2 d_{x1} & \theta_1^2 d_{y1} \\ \theta_2^2 d_{x2} & \theta_2^2 d_{y2} \end{pmatrix} \tag{39}$$

$$= \begin{pmatrix} \theta_1^2 d_{x1}^2 + \theta_2^2 d_{x2}^2 & \theta_1^2 d_{x1} d_{y1} + \theta_2^2 d_{x2} d_{y2} \\ \theta_1^2 d_{x1} d_{y1} + \theta_2^2 d_{x2} d_{y2} & \theta_2^2 d_{y2}^2 + \theta_1^2 d_{y1}^2 \end{pmatrix} \tag{40}$$

$$= \theta_0^2 \begin{pmatrix} d_{x1}^2 + d_{x2}^2 & d_{y1} d_{x1} + d_{x2} d_{y2} \\ d_{x1} d_{y1} + d_{x2} d_{y2} & d_{y2}^2 + d_{y1}^2 \end{pmatrix}. \tag{41}$$

The components of this matrix simplify to,

$$V_{r'_x r'_x} = \frac{\theta_0^2}{c_z^2 c_T^2} \left(c_y^2 + r_x^2 \right) \tag{42}$$

$$V_{r'_x r'_y} = \frac{\theta_0^2}{c_z^2 c_T^2} \left(-c_x c_y + r_x r_y \right) \tag{43}$$

$$V_{r'_y r'_y} = \frac{\theta_0^2}{c_z^2 c_T^2} \left(c_x^2 + r_y^2 \right) \tag{44}$$

In `ThinZPlaneMs` the expression for `Qxz` $= V_{r'_x r'_x}$ does reduce to the above form, but there is a typo in the expression for `Qyz` $= V_{r'_y r'_y}$; it reduces to:

$$V_{r'_y r'_y} = \frac{\theta_0^2}{c_z^2 c_T^2} \left(c_x^2 + r_x^2 \right). \tag{45}$$

The last term should be r_y^2, not r_x^2. There is also an error in the expression for `Qxy` $= V_{r'_x r'_y}$. The first term is,

$$-\theta_0^2 \frac{c_x c_y}{c_z^2} \tag{46}$$

but should be,

$$-\theta_0^2 \frac{c_x c_y}{c_T^2 c_z^2}. \tag{47}$$

The other error is in `ThinZPlaneMsSim`, in which the normalization factors of $1/c_T$ are missing from the definitions $\hat{v}$ and. $\hat{w}$.

Finally, there are several improper uses of `Math.pow(x,2.0)`:

```
trfxyp/ThinXYPlaneMs.java
trfeloss/DeDxBethe.java
trfzp/ThinZPlaneMs.java
```

Even better, the power function was used on:

```
pow( 1.  + cT*cT/cz/cz, 2.0) ,
```

which simplifies to $1/c_z^4$.