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Residuals With and Without the Local hit in the Fit

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Abstract

This note computes the error on the residual for two cases: that the the local hit is or is not included in the fit. The result is that $\sigma_r^2 = \sigma_m^2 \pm \sigma_f^2$, where σ_m is error on the local measurement and where σ_f is the error on the locally measured quantity, as predicted by the fit. The $+$ sign holds when the local hit is not included in the fit and the $-$ sign holds when the local hit is included in the fit. Excluding numerical precision considerations σ_r^2 is guaranteed to be non-negative.

1 Introduction

Consider that we have an array of silicon strip detectors which measure either x or y, which are arranged along the z axis and which are normal to the z axis. Figure 1 represents the situation at some plane which measures x. The point labelled x_1 represents the measurement of x made by the local device. The point labelled x_2 represents the predicted x, at this plane, which is made by obtained by a fit which uses all of the other planes in the device. In both cases the error bars represent the one σ errors, σ_1 and σ_2.

The points x_1 and x_2 represents two independent, uncorrelated measurements of x and, therefore, the optimal estimator of the x at which the track crossed this plane is obtained by the weighted mean of these two values. On the figure this point is denoted x_3, with error bars σ_3.

The weighted mean is computed in the usual fashion,

$$x_3 = \sigma_3^2 \left(\frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2} \right) \tag{1}$$

$$\sigma_3^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{2}$$

There are two ways to define the residual at this plane,

$$r = x_1 - x_2 \qquad \text{or} \qquad r = x_1 - x_3. \tag{3}$$

In both cases the residual is defined as $r = x_m - x_f$ but the cases differ in the meaning of x_f, where x_m refers to the local measurement and where x_m refers to the

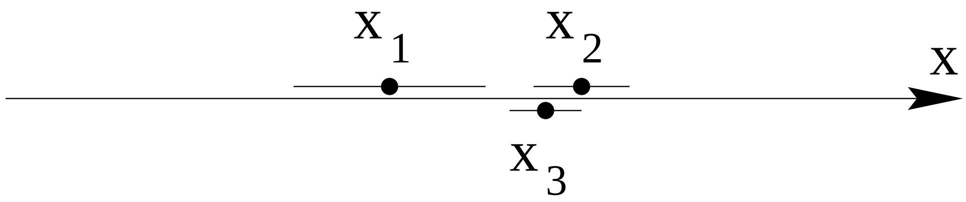

Figure 1: Schematic view of the status of a fit at some plane. The point x_1 represents the measurement made by the local device. The point x_2 represents the footprint, using all of the other hits on the track, of the track on the local device. The point x_3 represents the optimal measurement of the trajectory at this plane.

same quantity as predicted by the fit. In the first case we say that the local hit is not included in the fit and in the second case we say that the local hit is included in the fit.

Notice that the above discussion does not depend on what sort of trajectory is followed by the track, whether it is a straight line or a helix. All that is required is that that fitter be able to compute x_2 and σ_2. This is natural in the Kalman filter approach to track fitting.

2 Local Hit is Not Included in the Fit

When the local hit is NOT included in the fit, the two quantities in the definition of the residual are are independent, uncorrelated variables. Therefore the error on the residual is easy to compute,

$$\begin{aligned} \sigma_r^2 &= \left(\frac{\partial r}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial r}{\partial x_2}\right)^2 \sigma_2^2 \\ &= \sigma_1^2 + \sigma_2^2. \end{aligned} \quad (4)$$

Clearly σ_r^2 is positive for all sensible input values of σ_1 and σ_2.

3 Local Hit is Included in the Fit

When the hit is included in the fit the two quantities in the definition of the residual are no longer independent; that is x_3 depends on x_1. However the residual is still a function of the two independent, uncorrelated variables x_1 and x_2. Therefore,

$$
\begin{aligned}
\sigma_r^2 &= \left(\frac{\partial r}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial r}{\partial x_2}\right)^2 \sigma_2^2 \\
&= \left(1 - \frac{\partial x_3}{\partial x_1}\right)^2 \sigma_1^2 + \left(-\frac{\partial x_3}{\partial x_2}\right)^2 \sigma_2^2 \\
&= \left(1 - \frac{\sigma_3^2}{\sigma_1^2}\right)^2 \sigma_1^2 + \left(\frac{\sigma_3^2}{\sigma_2^2}\right)^2 \sigma_2^2 \qquad (5) \\
&= \sigma_1^2 - \sigma_3^2.
\end{aligned}
$$

Equation 2 guarantees that σ_3 is less than or equal to σ_1 and therefore, that σ_r is never negative. However numerical precision problems can sometimes cause σ_r to be either zero or negative; the classic example of precision problems is discussed in the next paragraph.

The case that $\sigma_3 = \sigma_1$ occurs in the limit that $\sigma_2 = \infty$. This condition will occur in a fit which, when all of the hits are included, has zero degrees of freedom. Examples include a straight line fit to 2 hits, a circle fit to 3 hits and a helix fit to 5 hits. In such a case, the residual $x_1 - x_2$ is has an infinite error and the residual $x_1 - x_3$ is identically zero, with zero error. This second residual is particularly prone to numerical precision problems.

4 Summary

The above discussion can be summarized in the statement,

$$\sigma_r^2 = \sigma_m^2 \pm \sigma_f^2. \qquad (6)$$

where σ_m is error on the local measurement and where σ_f is the error on the locally measured quantity, as predicted by the fit. The $+$ sign holds when the local hit is not included in the fit and the $-$ sign holds when the local hit is included in the fit.

In the limit of gaussian measurement errors, both definitions of the residual will produce a distribution of r/σ_r which is a unit gaussian.

This result holds for an arbitrary device which measures a single quantity and it can be generalized to devices which measure vector-valued quantities, such as pixel detectors.