Notes on Instantons

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Abstract

The instanton as a topological object in Yang-Mills gauge theory is presented.

Contents

1 Non-trivial field configurations

It often happens that the space of all possible field configurations may be given a non-trivial topology by the condition that some functional S of the fields is finite. This comes up in finding field configurations around which we may expand the field variables in path integrals in Euclideanized $d-$dimensional spacetime.

Examples of well-known extended field configurations correspond to:

- Skyrmions
- Domain boundaries
- Monopoles, vortex lines
- Instantons

We'll focus on the latter.

2 Path-integral approach

The most useful approach to the quantization of gauge theories appears to be Feynman's path integral method. From a geometric point of view, the path integral has the advantage of being able to take into account the global topology of the gauge potentials, while the canonical perturbation theory approach to quantization is sensitive only to the local topology.

At present mathematically precise theory of path-integration can be formulated only for spacetimes with positive signatures $(+,+,+,+)$, denoted Euclidean or *imaginary time* manifolds. Physically meaningful answers are obtainable by continuing the results [1] of the Euclidean path integration back to the Minkowski regime with signature $(-,+,+,+)$.

In the Euclidean path-integral approach to quantization, each field configuration $\phi(x)$ is weighted by the 'Boltzmann factor', *i.e.* the exponential of minus its Euclidean action, $\exp -S[\phi]$.

The complete generating functional for the various Green's functions of a theory is obtained by functionally integrating over all *inequivalent* field configurations. When performing this integration, one should naturally be careful not to include equivalent field configurations, *i.e.* related by (small) gauge transformations. It's needed then a measure that will count only once each gauge orbit, *i.e.* that be invariant under the action of the gauge group. [2]

Since the first-order functional variation of the action vanishes for solutions of the equations of motion, these configurations correspond to stationary points

[1] via Wick rotation

[2] such measures (mathematically) are called Haar measures, and the ansatz to proceed in the the context of Yang-Mills theories is known of course as the so-called Faddeev-Popov method

in the functional space. Therefore in the path integral approach we first seek solutions to the Euclidean field equations with minimum action and then compute quantum mechanical fluctuations around them.

3 Yang-Mills theory

A path integral is well defined only for Euclidean metric. To evaluate this integral it is important to find the local minima of the Euclidean action and compute the quantum fluctuations around them. The local minima of the $SU(2)$ gauge theory on a four-dimensional Euclidean space R^4 are called **instantons**.

3.1 Fields and the action

Consider the $SU(2)$ gauge theory defined on R^4, which is described by the principal bundle $P(R^4, SU(2))$.

The gauge potential [3] and strength are

$$\mathcal{A} = \mathcal{A}^a_\mu T_a dx^\mu$$

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \frac{1}{2}\ \mathcal{F}_{\mu\nu}\ dx^\mu \wedge dx^\nu$$

where

$$\mathcal{F}_{\mu\nu} \ = \ \partial_\mu\ \mathcal{A}_\nu - \partial_\nu\ \mathcal{A}_\mu \ + \ [\mathcal{A}_\mu, \mathcal{A}_\nu] = \mathcal{F}_{\mu\nu}{}^a T_a$$

$$\mathcal{F}_{\mu\nu}{}^a \ = \ \partial_\mu\ \mathcal{A}_{\nu a} - \partial_\nu\ \mathcal{A}_{\mu a} \ + \ \epsilon_{abc}\ \mathcal{A}_{\mu b}\ \mathcal{A}_{\nu c}$$

and $T_a = \frac{\sigma_a}{2i}$ are the $\mathcal{SU}(2)$ algebra generators

$$[T_a, T_b] = \epsilon_{abc} T_c$$

The Bianchi identity is

$$\mathcal{D}\ \mathcal{F} \ \equiv \ d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$$

The action for Yang-Mills theories is

$$S_{YM}[\mathcal{A}] = -\frac{1}{4}\int_M \mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} g^{\frac{1}{2}}\ d^4x \ = \ \frac{1}{2}\int_M Tr\ \mathcal{F} \wedge *\mathcal{F}$$

[3] we do not include here an index labelling the local trivialization used, as we can take a single one, *i.e.* R^4 is trivial

while in the Euclideanized form [4]

$$S^E_{YM}[\mathcal{A}] = +\frac{1}{4}\int_M \mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu}g^{\frac{1}{2}}\ d^4x \ = \ -\frac{1}{2}\int_M Tr\ \mathcal{F}\wedge *\mathcal{F}$$

which is positive definite. [5]

The Yang-Mills equations found by varying the action with respect to $\mathcal{A}_\mu$ may be written as

$$\mathcal{D}*\mathcal{F} = 0, \quad or \quad \mathcal{D}_\mu\mathcal{F}^{\mu\nu} = 0$$

$$d*\mathcal{F} + A\wedge *\mathcal{F} - *\mathcal{F}\wedge A = 0$$

while the Bianchi identity is as above

$$d\mathcal{F} + A\wedge\mathcal{F} - \mathcal{F}\wedge A = 0$$

3.2 The (anti-) self-dual fields and the lower bound of the action

In order to find the minimum of the action configurations of the theory, let's consider the inequality [6]

$$\int_M tr\{\mathcal{F}_{\mu\nu} \pm *\mathcal{F}_{\mu\nu}\}^2\ d^4x \geq 0$$

This bound is saturated by the (anti-) self-dual [7]field configurations

$$\mathcal{F} = \mp * \mathcal{F}$$

which solve the field equations [8] now implied by the Bianchi identities

$$\mathcal{D}_\mu\mathcal{F}_{\mu\nu} = \mathcal{D}_\mu * \mathcal{F}_{\mu\nu} = 0$$

Noting that $\mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu} \ = \ \mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}$, the previous inequality may be written as

$$\int_M tr\{2\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} \pm 2\mathcal{F}_{\mu\nu} * \mathcal{F}_{\mu\nu}\}\ d^4x \geq 0$$

[4]The Hodge * is now taken with respect to the Euclidean metric

[5]The contribution of each gauge potential or connection $\mathcal{A}_\mu(x)$ to the path integral is thus bounded and well-behaved.

[6]Bogomol'nyi inequality

[7]*Note* : $*\mathcal{F} = \frac{1}{2} * \mathcal{F}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ with $*\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}$; also $tr\mathcal{F}\wedge\mathcal{F} = -\frac{1}{2}tr\mathcal{F}_{\mu\nu} * \mathcal{F}_{\mu\nu}dx^1 \wedge \ldots \wedge dx^4$

[8]note however that instead this equality is a first order differential equation, whereas the Yang-Mills field equations are second order

The action now becomes

$$S = -\frac{1}{2}\int_M Tr\ \mathcal{F}\wedge *\mathcal{F} = \mp\frac{1}{2}\int_M Tr\ \mathcal{F}\wedge\mathcal{F}$$

which is proportional, $S = 4\pi^2|W[\mathcal{A}]|$, to the integral of the second Chern class

$$W[\mathcal{A}] = -\frac{1}{8\pi^2}\int_M Tr\ \mathcal{F}\wedge\mathcal{F}$$

't Hooft called such special field configurations instantons.

3.3 Finite action

Yang-Mills instantons are finite action solutions to the Yang-Mills equations of motion.

The requirement that the action [9]

$$S = -\frac{1}{2}\int_M tr\ \mathcal{F}\wedge\mathcal{F}$$

be finite imposes that at infinity $S = 0$, which happens iff $\mathcal{F}_{\mu\nu} = 0$ there. Also note that this is a gauge invariant condition. Indeed, under a gauge transformation g,

$$\mathcal{A}_\mu \to g\mathcal{A}_\mu g^{-1} + g\ \partial_\mu g^{-1}$$

$$\mathcal{F}_{\mu\nu} \to g\mathcal{F}_{\mu\nu} g^{-1}$$

Therefore, it will be satisfied not only by $\mathcal{A}_\mu = 0$ but by any gauge-transformed field (called **pure gauge**) obtained from it,

$$\mathcal{A}_\mu(x) \to g(x)\ \partial_\mu g^{-1}(x) \qquad as\ \ |x| \to \infty$$

The spacetime infinity can be identified with S^3. This way, it is defined a map $g : S^3 \to G$ which is classified by $\pi_3(G)$. In the case of interest here, $G \cong SU(2)$, we have $\pi_3(SU(2)) \cong Z$.

It is convenient to compactify the spacetime to the S^4 sphere, obtained by adding the infinity to R^4. This manifold is described by two patches, $S^4 = U_+ \cup U_-$, where $U_+ \cap U_-$ is homeomorphic to the S^3 sphere. (U_+ may represent the standard spacetime, the equatorial S^3 sphere the spacetime infinity, and U_- just an artifact of the compactification).

[9]Please note that $S = -\frac{1}{2}\int_M d^dx\ tr\ \mathcal{F}\wedge\mathcal{F}$ can either be regarded as the action for quantum gauge fields in a Euclidean d-dimensional spacetime, or the potential energy for classical gauge fields in temporal gauge, with $\mathcal{A}^0_\alpha = 0$, in **(d+1)**-dimensional spacetime

Consider $\mathcal{A}_-(x) = 0$, then all the topological information about the bundle is contained in the transition function $g_{+-}(x)$ for $x \in U_+ \cap U_-$,

$$\mathcal{A}_+ = g_{+-}^{-1} dg_{+-}$$

The map g above is then just the transition function $g_{+-}(x) : S^3 \to SU(2)$.

3.4 Compactifiability of finite-action Yang-Mills connections

Consider a local form $\mathcal{A}(x)$ of the connection one-form on a manifold $\hat{M}$ which is a compact manifold M lacking a point, $\hat{M} = M - \{0\}$, with the corresponding local form of the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ being harmonic. Then all Euclidean finite action Yang-Mills solution over $\hat{M}$ can be smoothly extended to the compact manifold M.

This theorem tells that any self-dual finite-action solution to the Euclidean Yang-Mills equations must describe a bundle with a compactified spacetime base manifold.

4 Instantons

Consider again gauge configurations [10] of the form $\mathcal{A}_- = 0$ and $\mathcal{A}_+ = g_{+-}^{-1} dg_{+-}$. Self-dual solutions of this kind are named instantons.

One example of such solution is provided by referring to $SU(2)$. Actually, this case is rather special as it can be identified with the $SU(2)$ subgroup of any larger group; indeed, this is sufficient for $SU(N)$, $N \geq 2$.

Maps $g : S^3 \to SU(2)$ are classified [11] according to $\pi_3(S^3) = Z$. Representatives of each homotopy class can be provided by the following maps.

- (0) The constant map g_0 specifies the 3rd homotopy trivial class of SU(2)

$$g_0 : x \mapsto e$$

- (1) The identity g_1 map specifies the 3rd homotopy class 1 of SU(2)

$$g_1 : x \mapsto \frac{1}{r}[x^4 1 + i x^i \sigma_i], \quad r^2 = (x^4)^2 + \mathbf{x}^2$$

- (n) The map $(g_1)^n$ specifies the 3rd homotopy class n of SU(2)

$$g_n \equiv (g_1)^n : x \mapsto \frac{1}{r^n}[x^4 1 + i x^i \sigma_i]^n, \quad r^2 = (x^4)^2 + \mathbf{x}^2$$

[10] These are also particularly relevant when dealing with the strong CP problem

[11] Note that $SU(2) \cong S^3$ through $x^4 1 + i x^i \sigma_i \in SU(2) \quad \leftrightarrow \quad (x^4)^2 + \mathbf{x}^2 = 1$

Consider the homotopy class 1, and take the identity map $g \equiv g_1$. The corresponding gauge potential [12] is

$$\mathcal{A}_\mu(x) = \frac{r^2}{r^2+\lambda^2} g^{-1}(x)\partial_\mu g(x)$$

and the corresponding field strength is

$$\mathcal{F}_{\mu\nu} = \frac{4\lambda^2}{r^2+\lambda^2}\sigma_{\mu\nu}$$

where $\sigma_{ij} \equiv [\sigma_i, \sigma_j], \quad \sigma_{i0} \equiv \frac{1}{2}\sigma_i \equiv -\sigma_{0i}$.
The explicit forms for these are

$$\mathcal{A}_\mu(x) = -2i\Sigma_{\mu\nu}\frac{x^\nu}{r^2+\lambda^2}$$

and

$$\mathcal{F}_{\mu\nu} = 4i\Sigma_{\mu\nu}\frac{\lambda^2}{[r^2+\lambda^2]^2}$$

where $\Sigma_{\mu\nu} = \eta^{i\mu\nu}\frac{\sigma^i}{2}$ for $i = 1,2,3$, $\eta^{i\mu\nu} = -\eta^{i\nu\mu} = \epsilon^{i\mu\nu}$ for $\mu, \nu = 1,2,3$ and $\eta^{i\mu 4} = \delta^{i\mu}$.

Some remarks: the tensor $\Sigma_{\mu\nu}$ is antisymmetric and self dual, so is $\mathcal{F}_{\mu\nu}$, as it should; it is indeed a solution of the Euclidean Yang-Mills equations of motion which on the sphere S^3 correspondent to $|x| \to \infty$ is a pure gauge; the winding number of this field is one, as is the homotopy class label of the correspondent g defined on S^3.

5 Topological properties

Notice [13] that $tr\mathcal{F}^2$ is a closed form,

$$d(tr\mathcal{F}^2) \;=\; tr[d\mathcal{F}\;\mathcal{F} + \mathcal{F}\;d\mathcal{F}] \;=\; tr\{-[\mathcal{A},\mathcal{F}]\mathcal{F} - \mathcal{F}[\mathcal{A},\mathcal{F}]\} \;=\; 0$$

where the Bianchi identity $d\mathcal{F} + [\mathcal{A},\mathcal{F}] = 0$ has been used. Then, by Poincaré's lemma, $tr\mathcal{F}^2$ is locally exact; *i.e.*, there is a local 3−form $Q_3[\mathcal{A}]$ such that

$$tr\mathcal{F}^2 \;=\; dQ_3[\mathcal{A}]$$

in every local trivialization.

$Q_3[\mathcal{A}]$ is the **Chern-Simons form** (or **or 0-cochain**)

[12] λ is to be interpreted as the size of the instanton

[13] Here for simplification of notation we omit the wedge product and write *e.g.* $\mathcal{F}\wedge\mathcal{F} \equiv \mathcal{F}^2$

$$Q_3[\mathcal{A}] \;=\; tr\;(\;\mathcal{A}\;d\mathcal{A}\;+\;\frac{2}{3}\mathcal{A}^3)$$

$tr\mathcal{F}^2$ is the second **Chern character** (or **Chern-Pontrjagin density**);in case $tr\mathcal{F}^2$ is considered over S^4, it is an element of the de Rham cohomology group $H^4(S^4)$. It is clearly gauge invariant.

Using the parameterization of $S^4 = U_- \cup U_+$ introduces before, and $\mathcal{A}_- = 0$, $\mathcal{A}_+ = g_{+-}^{-1} dg_{+-}$, then this gives for $W[\mathcal{A}]$,

$$\begin{aligned}
W[\mathcal{A}] &= \tfrac{-1}{8\pi^2} \int_{U_+\cup U_-} tr\;\mathcal{F}^2 \\
&= \tfrac{-1}{8\pi^2}\;(\;\int_{U_+} dQ_3[\mathcal{A}_+]\;+\;\int_{U_-} dQ_3[\mathcal{A}_-]\;) \\
&= \tfrac{-1}{8\pi^2}\;(\;\int_{\partial U_+} Q_3[\mathcal{A}_+]\;+\;\int_{\partial U_-} Q_3[\mathcal{A}_-]\;) \\
&= \tfrac{-1}{8\pi^2} \int_{S^3} (\;Q_3[\mathcal{A}_+]\;-\;Q_3[\mathcal{A}_-]\;) \\
&= \tfrac{-1}{8\pi^2} \int_{S^3} Q_3[g_{+-}^{-1} dg_{+-}] \\
&= \tfrac{1}{24\pi^2} \int_{S^3} tr\;(g_{+-}^{-1} dg_{+-})^3
\end{aligned}$$

where it has been used Stoke's theorem, and noted that $\partial U_+ = -\partial U_- \sim S^3$. [14]

$W[\mathcal{A}]$ is related with the homotopy group $\pi_3(G)$. [15] For the case $G = SU(n)$, $(n \geq 2)$, $\pi_3(SU(n)) = Z$; in this case, $W[\mathcal{A}] \in Z$ is the winding number, labelling the maps g_{+-}. This is shown explicitly next.

Take the representatives g_n of the elements of $\pi_3(S^3)$ considered in the last section:

- (0) For the constant map $g_0(x) \equiv e$, $\mathcal{A} = 0$ on S^3, and as the bundle is trivial $\mathcal{A} = 0$ on S^4 also; then $W[\mathcal{A}] = 0$
- (1) For the identity map $g_1 : x \mapsto \frac{1}{r}[x^4 1 + i x^i \sigma_i]$,

  $$\mathcal{A} = \frac{r^2}{r^2 + \lambda^2} g^{-1}(x)\;dg(x)$$

  and it is shown that

  $$W[\mathcal{A}] = 1$$

- (n) It can be shown also, please see the discussion of the Maurer-Cartan integral invariant that $g_2 = g_1 g_1$ has winding number 2; in general then

  $$W[\mathcal{A}] = n$$

[14] Note that we have recovered the form of the winding number in terms of the Maurer-Cartan invariant integral

[15] and with the Dirac operator