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Volume : 37

Issue :

Date : 1966

Pages : 2568-eoa

Author : Beth, Richard A.

Title : Complex Representation and Computation of Two-Dimensional Magnetic Fields

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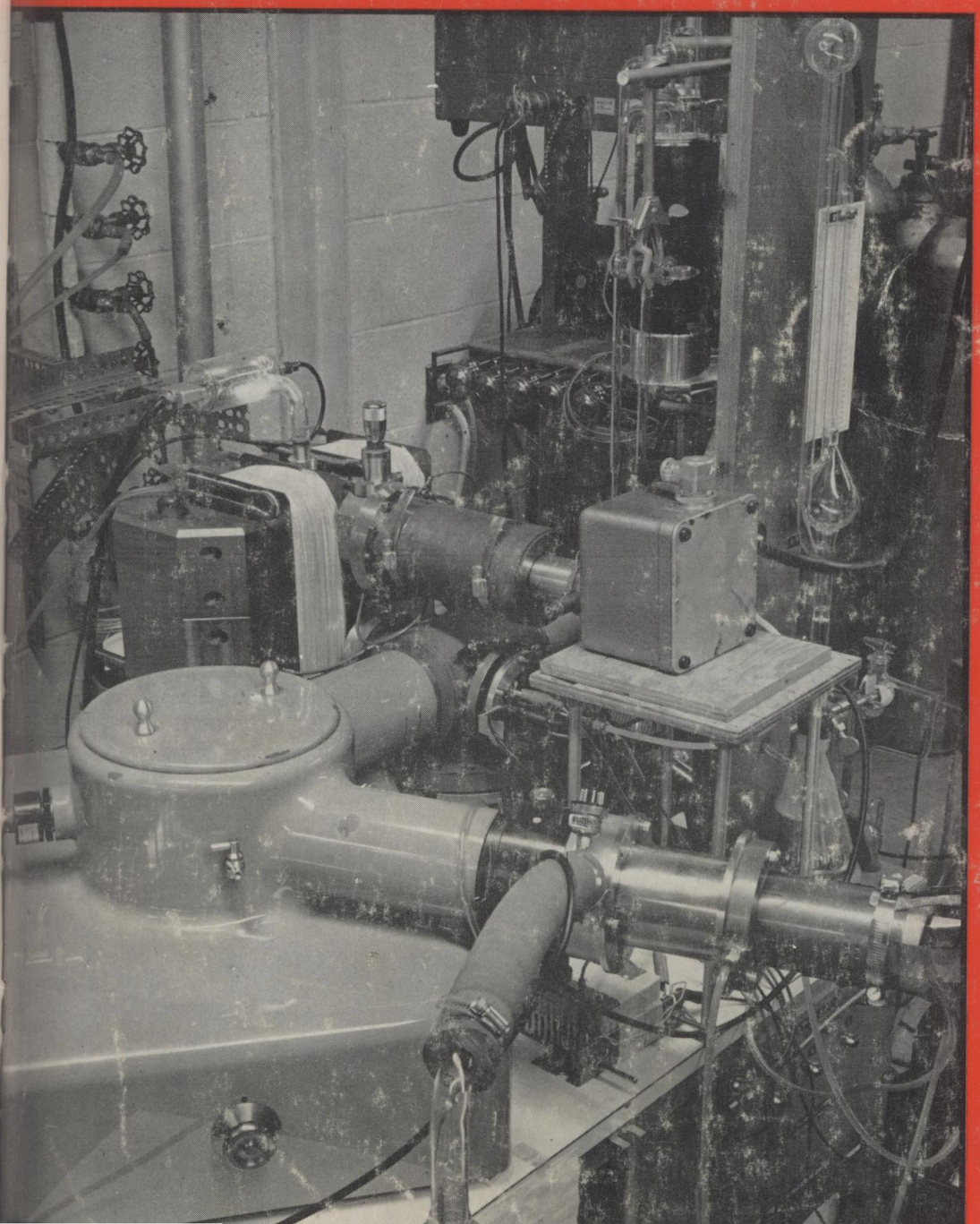

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The *Journal of Applied Physics* is published monthly at Prince and Lemon Streets, Lancaster, Pennsylvania 17604.
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Complex Representation and Computation of Two-Dimensional Magnetic Fields*

RICHARD A. BETH
Brookhaven National Laboratory, Upton, New York
(Received 20 January 1966)

The method of evaluating two-dimensional fields as analytic functions of a complex variable without resort to absolute values and complex conjugates is described and applied to the magnetic field produced by straight currents. Explicit formulas are derived suitable for numerical computation of vector and scalar potentials, of field components, and of field gradients for round and polygonal (including rectangular) conductors, current ribbons, and arrays thereof.

THE magnetic field in a simply connected region excluding currents and permeable materials can be described either by a scalar potential Φ or by a vector potential $\mathbf{A}$.

In the two-dimensional case it is advantageous to combine the two descriptions into a single complex representation. The method is here applied to the evaluation of the magnetic field due to an infinite straight conductor of arbitrary cross section. The resulting *complex* potential, field, and gradient expressions for polygonal or rectangular conductors and current ribbons are given in integrated form suitable for numerical computation.

COMPLEX REPRESENTATION

Consider a two-dimensional magnetic field parallel to the X, Y plane. The field strength components can be written in terms of a scalar potential $\Phi=\Phi(X,Y)$:

$$H_X=H_X(X,Y)=-\partial\Phi/\partial X,$$
$$H_Y=H_Y(X,Y)=-\partial\Phi/\partial Y. \tag{1}$$

For a two-dimensional field the only non-constant component of the vector potential is the component perpendicular to the field plane, which we write $A=A(X,Y)$, so that

$$B_X=\mu H_X=\partial A/\partial Y,\quad B_Y=\mu H_Y=-\partial A/\partial X. \tag{2}$$

For unit permeability, $\mu=1$, we have from (1) and (2)

$$\partial A/\partial X=\partial\Phi/\partial Y,\quad \partial A/\partial Y=-\partial\Phi/\partial X.$$

Regarded as Cauchy–Riemann equations these relations are the necessary and sufficient conditions that $A+i\Phi$ be an analytic function of the complex variable

$$Z=X+iY.$$

We note that $\Phi+iA$ is an analytic function of $Z^*=X-iY$ but not of Z. Any analytic function of $A+i\Phi$ is again an analytic function of Z. We choose to call

$$W=W(Z)=-(A+i\Phi)+\text{const}, \tag{3a}$$

the complex *potential* of the two-dimensional field; any arbitrary constant may be added to W without changing the field quantities derived from it.

Derivatives of W are, in turn, analytic functions of Z. The first derivative of W,

$$\frac{dW}{dZ}=-\frac{\partial A}{\partial X}-i\frac{\partial\Phi}{\partial X}=i\frac{\partial A}{\partial Y}-\frac{\partial\Phi}{\partial Y},$$

is the complex *field* [cf. Eqs. (1) and (2)]

$$dW/dZ=H=H(Z)=H_Y(X,Y)+iH_X(X,Y), \tag{3b}$$

where the field components H_Y and H_X appear as real functions of X and Y; the Cauchy–Riemann relations between them express the field equations curl $\mathbf{H}=0$ and div $\mathbf{B}=0$ for this two-dimensional case.

It is important to notice that H_X+iH_Y is *not* an analytic function of Z. We emphasize the choice of the combination $A+i\Phi$ instead of $\Phi+iA$ in Eq. (3a). This choice is made to preserve the full advantage of analyticity without resort to complex conjugates or absolute values in the physical interpretation of derivatives of $W(Z)$.[1] The derivations that follow illustrate the nature of this advantage.

The second derivative of W is the complex *gradient*:

$$\frac{d^2W}{dZ^2}=G=G(Z)=\frac{\partial H_Y}{\partial X}+i\frac{\partial H_X}{\partial X}, \tag{3c}$$

where $\partial H_Y/\partial X(=\partial H_X/\partial Y)$ is the transverse gradient and $\partial H_X/\partial X(=-\partial H_Y/\partial Y)$ is a measure of the longitudinal gradient.

Higher field "gradients" can be found as analytic functions of Z by further differentiation. At each stage the field conditions are expressed by the fact that each higher gradient comprises just two real functions of X and Y which satisfy Cauchy–Riemann relations.

THE FIELD DUE TO A CURRENT FILAMENT

An infinitely long straight filament current I flowing upward (in the usual right handed+sense) perpendicular to the X, Y plane at $Z=0$ has the complex potential

$$W=2I\log Z+\text{const}=-A-i\Phi, \tag{4}$$

* Work carried out under the auspices of the U. S. Atomic Energy Commission.

[1] For example, see G. Wendt, in *Handbuch der Physik*, S. Flügge, Ed. (Springer-Verlag, Berlin, 1958), Vol. XVI, p. 57, Eq. (49.6); E. Weber, *Electromagnetic Fields* (John Wiley & Sons, Inc., New York, 1950), p. 283, Eq. (15); J. Van Bladel, *Electromagnetic Fields* (McGraw-Hill Book Co., Inc., New York, 1964), p. 136, Eq. (5.49).

as may be verified by differentiating with respect to Z; according to (3b),

$$\frac{dW}{dZ}=H_Y+iH_X=\frac{2I}{Z}=\frac{2I}{X+iY},$$

which yields the usual formulas:

$$H_Y=2IX/(X^2+Y^2) \quad \text{and} \quad H_X=-2IY/(X^2+Y^2).$$

The potential (4) and its derivatives obviously also apply to the field outside a conductor or wire of circular cross section.

In (4) $Z=X+iY$ appears as the pure number ratio of Z to the unit of length used in measuring X and Y; any change of the unit of length merely appears as an additive constant in the logarithm and may be absorbed into the arbitrary constant. However, the unit of length used does affect the derivatives (3b) and (c). For Gaussian units X and Y are in centimeters and I in abamperes (1 abamp=10 A).

If we write $Z=X+iY=Re^{i\theta}$ then $\log Z=\log_e R+i\theta$ and θ (radians) increases by 2π when the point Z is taken once around the origin in the positive direction. Thus the scalar potential Φ in (4) decreases by $4\pi I$ when the field point is taken once around the current I in the positive direction exactly as required by the field equations. The restriction to simply connected regions is identical with the restriction to one branch of the complex logarithm of Z.

CONDUCTOR OF ARBITRARY CROSS SECTION

Let S in Fig. 1 be a normal cross section of an infinitely long straight uniform conductor. Let the current density in S be $\sigma=\sigma(x,y)$ abamp/cm^2.

The potential at a field point Z is the integral over the region S of filament contributions like (4),

$$dW=2\sigma(x,y)\log(Z-z)dxdy+\text{const},$$

where small $z=x+iy$ is used to distinguish the *source point* (filament position) from the capital $Z=X+iY$ used for the *field point*.

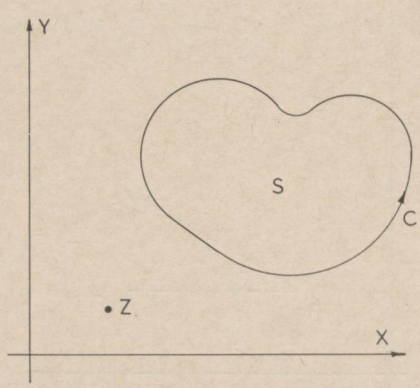

FIG. 1. Intersection of conductor of arbitrary cross section with complex plane $Z=X+iY$.

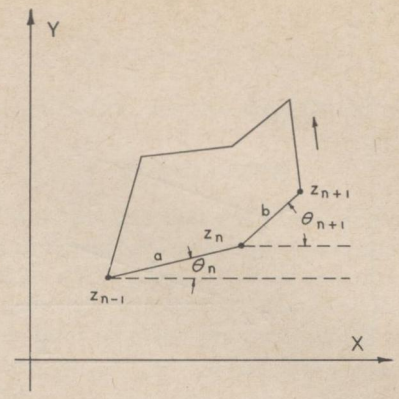

FIG. 2. Cross section of polygonal conductor.

It is convenient to reverse the sign of $Z-z$ [assimilating $\log(-1)$ into the arbitrary constant] and use

$$r=z-Z=(x-X)+i(y-Y),$$

as the variable of integration. Thus the potential of the conductor may be written in the form

$$W=W(Z)=2\int\int_S \sigma(x,y)\log r dxdy+\text{const}, \tag{5}$$

where Z is a fixed parameter during the integration.

For $\sigma=I/S=\text{const}$, we can apply the general integral transformation

$$\int\int_S \frac{\partial P}{\partial x}dxdy=\oint_C Pdy \tag{6}$$

with $P=r\log r$ to the surface integral in (5) and obtain (since $\oint dy=0$) the line integral

$$W(Z)=2\sigma\oint r\log r dy+\text{const}=-A-i\Phi, \tag{7a}$$

taken in the positive sense around the boundary C of the conductor cross section S in Fig. 1. The field and gradient functions are obtained according to (3b) and (3c) by differentiating under the integral with respect to the parameter Z; thus we derive the general formulas:

$$H(Z)=-2\sigma\oint \log r dy=H_Y+iH_X, \tag{7b}$$

and

$$G(Z)=2\sigma\oint dy/r, \tag{7c}$$

for the magnetic field of a conductor of arbitrary cross section.

POLYGONAL CONDUCTOR

If the normal cross section of the conductor is a polygon as shown in Fig. 2, explicit expressions for the field quantities (7) can be given in terms of the coordi-

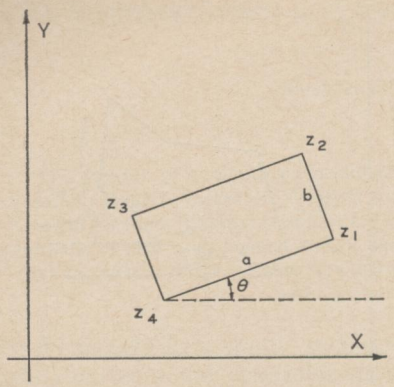

FIG. 3. General position and orientation of rectangular conductor.

nates x_n and y_n of the N vertices, $z_n = x_n + iy_n$, numbered $n=1, 2\cdots N$ in the positive direction around the polygon.

We introduce, as complex functions of $r=z-Z$:

$$w(r)=\tfrac{1}{2}r^2 \log r, \tag{8a}$$

$$h(r)=r \log r, \tag{8b}$$

$$g(r)=\log r. \tag{8c}$$

Then the expressions (7a,b,c) can be written in the forms

$$W(Z)=2\sigma \oint (dw/dr)dy+\text{const}, \tag{9a}$$

$$H(Z)=-2\sigma \oint (dh/dr)dy, \tag{9b}$$

$$G(Z)=2\sigma \oint (dg/dr)dy, \tag{9c}$$

since $\oint dy=0$ and

$$\oint rdy=\oint zdy=S=\text{const}. \tag{10}$$

On a straight path, like the polygon edge of length a from z_{n-1} to z_n in Fig. 2, dy is proportional to dz. Also, since Z is a fixed parameter during the integration, $dz=dr$. Thus, for the edge from vertex $n-1$ to vertex n, we may set

$$dy=\alpha_n dr,$$

where α_n is the complex constant

$$\alpha_n=\frac{y_n-y_{n-1}}{z_n-z_{n-1}}=\frac{a\sin\theta_n}{ae^{i\theta_n}}=e^{-i\theta_n}\sin\theta_n \tag{11}$$

which is seen to be independent of the length a of the side. Thus we obtain from (9)

$$W(Z)=2\sigma \sum \alpha_n(w_n-w_{n-1})+\text{const}, \tag{12a}$$

$$H(Z)=-2\sigma \sum \alpha_n(h_n-h_{n-1}), \tag{12b}$$

$$G(Z)=2\sigma \sum \alpha_n(g_n-g_{n-1}), \tag{12c}$$

where each sum is taken over all vertices, $n=1$ to N, $r_n=z_n-Z$, and $w_n=w(r_n)$, etc., in (8). If we combine the two terms arising from each vertex and define

$$\beta_n=\alpha_{n+1}-\alpha_n=e^{-i(\theta_{n+1}+\theta_n)}\sin(\theta_{n+1}-\theta_n), \tag{13}$$

we can write (12) in the form

$$W(Z)=-2\sigma \sum \beta_n w_n+\text{const}, \tag{14a}$$

$$H(Z)=2\sigma \sum \beta_n h_n, \tag{14b}$$

$$G(Z)=-2\sigma \sum \beta_n g_n, \tag{14c}$$

to describe the field of a conductor of polygonal cross section.

EQUIANGULAR POLYGON

For a polygon in which all the angles (but not necessarily the lengths of the sides) are the same, these expressions may be further simplified. Let

$$\theta_{n+1}-\theta_n=\Delta=2\pi/N,$$

be the constant exterior angle between successive sides and let the side from z_N to z_1 make the angle $\theta_1=\theta$ with the positive X axis. Then $\theta_n=\theta+(n-1)\Delta$ and we have for the coefficients in (14)

$$2\sigma\beta_n=K\delta^{n-1},$$

where

$$K=2\sigma e^{-i(2\theta+\Delta)}\sin\Delta=2\sigma\beta_1 \tag{15a}$$

and

$$\delta=e^{-i2\Delta}=e^{-i4\pi/N}. \tag{15b}$$

Hence

$$W(Z)=-K\sum \delta^{n-1}w_n+\text{const}, \tag{16a}$$

$$H(Z)=K\sum \delta^{n-1}h_n, \tag{16b}$$

$$G(Z)=-K\sum \delta^{n-1}g_n, \tag{16c}$$

describe the field of a conductor whose cross section is a constant angle polygon.

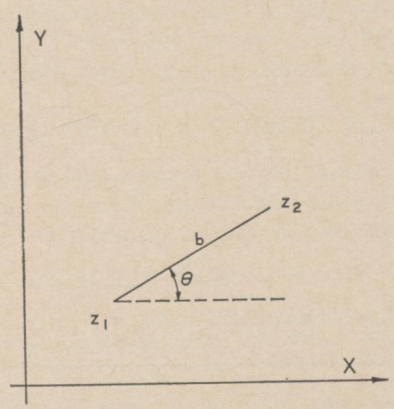

FIG. 4. Flat current ribbon.

RECTANGULAR CONDUCTOR

For a rectangular conductor with sides a and b as shown in Fig. 3, $N=4, \Delta=\pi/2, \delta=-1$, and $\sin\Delta=1$. The coefficients in (14) may be simplified as follows:

$$2\sigma\beta_n=(-1)^{n-1}K,$$

where

$$K=-2\sigma i e^{-i2\theta}=2I/(z_1-z_4)(z_2-z_1) \tag{17}$$

since $(z_1-z_4)=ae^{i\theta}$, $(z_2-z_1)=be^{i(\theta+\pi/2)}$, and $I=\sigma ab$ =total current. Hence we have, for a rectangular conductor

$$W(Z)=-K[w_1-w_2+w_3-w_4]+\text{const}, \tag{18a}$$

$$H(Z)=K[h_1-h_2+h_3-h_4], \tag{18b}$$

$$G(Z)=-K[g_1-g_2+g_3-g_4] =K\log\frac{(z_2-Z)(z_4-Z)}{(z_1-Z)(z_3-Z)}. \tag{18c}$$

CURRENT RIBBON

A current sheet composed of straight current filaments perpendicular to the X, Y plane will produce a complex potential

$$W(Z)=2\int_C \lambda(s)\log r ds+\text{const}, \tag{19}$$

where the integration is carried along the curve C, in which the sheet intersects the X, Y plane, from one edge of the sheet to the other and $\lambda(a)$ abamp/cm is the linear current density at the real distance s measured along C from the initial point.

For a flat current ribbon of breadth b with uniform current distribution, $\lambda=I/b=$constant. The integration is carried along the straight line from z_1 to z_2 in Fig. 4 so that

$$ds/dr=b/(z_2-z_1)=e^{-i\theta}.$$

Hence (19) can again be integrated explicitly to give

$$W(Z)=-k(h_1-h_2)+\text{const}, \tag{20a}$$

$$H(Z)=k(g_1-g_2)=k\log[(z_1-Z)/(z_2-Z)], \tag{20b}$$

$$G(Z)=-k\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=k\left[\frac{1}{z_2-Z}-\frac{1}{z_1-Z}\right], \tag{20c}$$

where

$$k=2\lambda e^{-i\theta}=2I/(z_2-z_1) \tag{21}$$

for the magnetic field due to a straight flat ribbon.

CONCLUSION

Any of the formulas that have been derived may be separated into real and imaginary parts for numerical computation of the components of potential, field, or gradient according to Eqs. (3). However, the formulas in complex form as given are particularly suitable for use with a computer that can calculate complex quantities directly.

The calculation of arrays of conductors amounts, by superposition, to simple cumulative addition in the computer. Arrays may include filaments or round wires, polygonal and rectangular conductors, round or polygonal holes in conductors (by subtraction), and flat current ribbons. Each conductor is specified by a signed current or current density and appropriate z values for its position in the complex plane. The computer program must take account of the fact that the computer will evaluate only the principal value of each complex logarithm, e.g., a nonnegative imaginary part from 0 to 2π.

Field or gradient components can be computed directly without prior evaluation of potentials.

The formulas for the vector potential of a rectangular conductor centered at the origin as previously given by Stevenson and Park[2] and by Strutt[3] correspond to the real part of Eq. (18a) for the special case, $\theta=0$.

The complex formulation for two-dimensional fields adopted in this paper is applicable to other problems.[4] The potential function $W(Z)$ corresponds to the mapping function used in Schwarz–Christoffel methods.

ACKNOWLEDGMENT

I wish to thank J. P. Blewett and E. D. Courant for interesting discussions.

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