Lecture 1
or
Nonlinear oscillations

Tim Zolkin, Sergei Nagaitsev

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zolkin@fnal.gov

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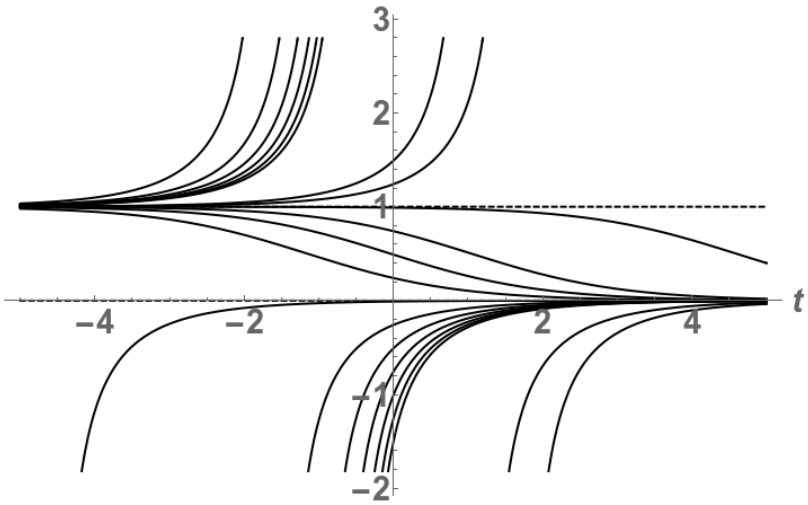
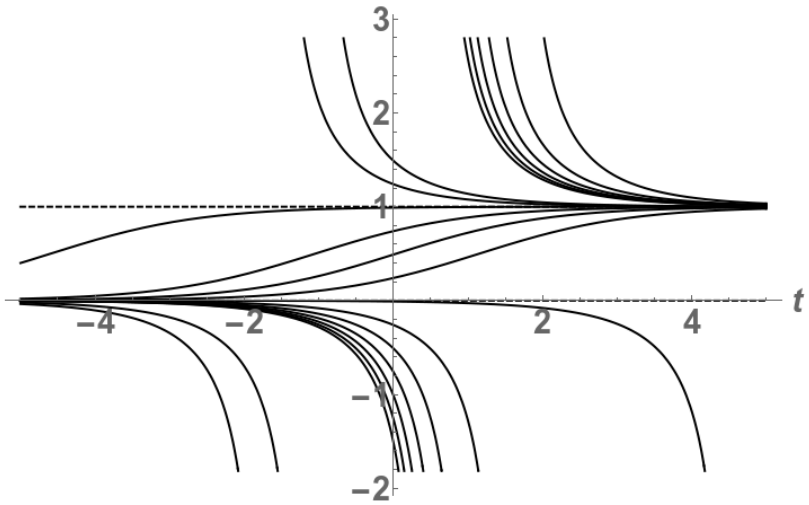
Part I

Continuous Dynamics

0. Logistuc equation, $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = r\,x\,(1-x)$

$$x(t) = \frac{1}{1 + \left(1 + \frac{1}{x_0} - 1\right)\,e^{-r\,t}}$$





A.1 Elliptic integrals

The function $f(x)$ is called *elliptic integral* if it can be expressed in the form

$$f(x) = \int_{x_0}^{x} \mathcal{R}\left(t, \sqrt{\mathcal{P}(t)}\right) \mathrm{d}t,$$

where x_0 is a constant, $\mathcal{R}$ is a rational function of t and of the square root of polynomial function of degree 3 or 4 with no multiple roots

$$\mathcal{P}(t) = a_0 t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4, \qquad a_0 \neq 0 \vee a_1 \neq 0.$$

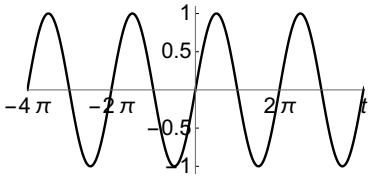
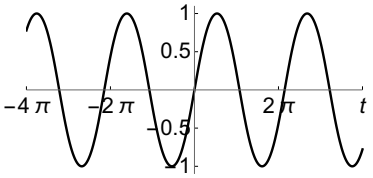
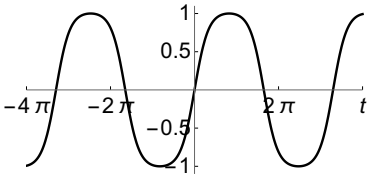
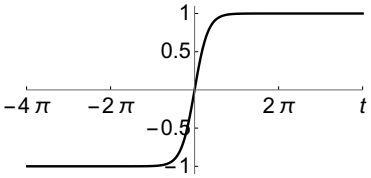
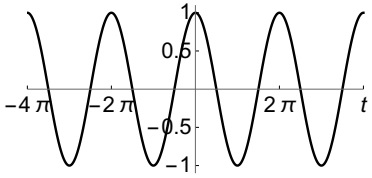
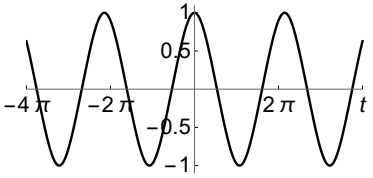
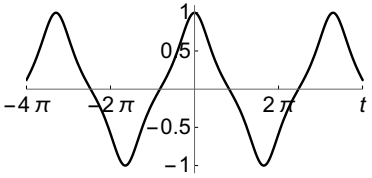
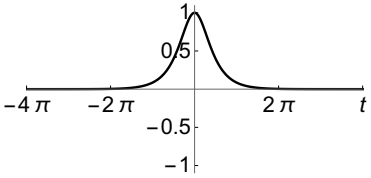
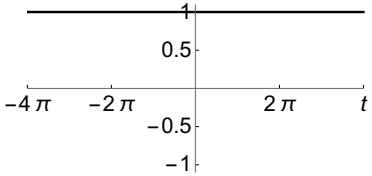
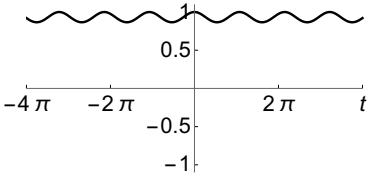
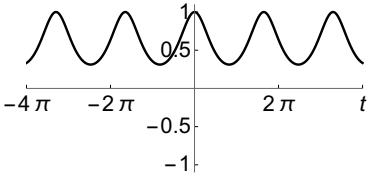
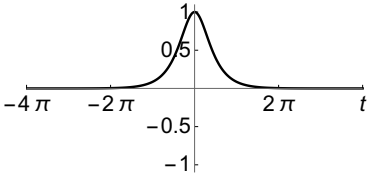
In the case when polynomial function has repeated roots the integrals are called *pseudoelliptic* and can be expressed via finite number of elementary functions (algebraic, exponential, logarithmic, trigonometric and inverse trigonometric). All elliptic integrals can be expressed as a linear combination of elementary functions and canonical set of elliptic integrals.

A.2 Elliptic integrals and Jacobi elliptic functions

$$
\begin{aligned}
\mathrm{F}(t,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(1-k^2\,\tau^2)}} \\
\mathrm{E}(t,k) &= \int_0^t \sqrt{\frac{k'^2+k^2\,\tau^2}{1-\tau^2}}\,\mathrm{d}\,\tau \\
\Pi(t,\alpha^2,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{(1-\alpha^2\tau^2)\sqrt{(1-\tau^2)(\tau^2-k'^2)}} \\
\mathrm{sn}^{-1}(t,k) &= \int_0^t \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(1-k^2\,\tau^2)}} \\
\mathrm{cn}^{-1}(t,k) &= \int_t^1 \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(k'^2+k^2\,\tau^2)}} \qquad k' = \sqrt{1-k^2} \\
\mathrm{dn}^{-1}(t,k) &= \int_t^1 \frac{\mathrm{d}\,\tau}{\sqrt{(1-\tau^2)(\tau^2-k'^2)}}
\end{aligned}
$$

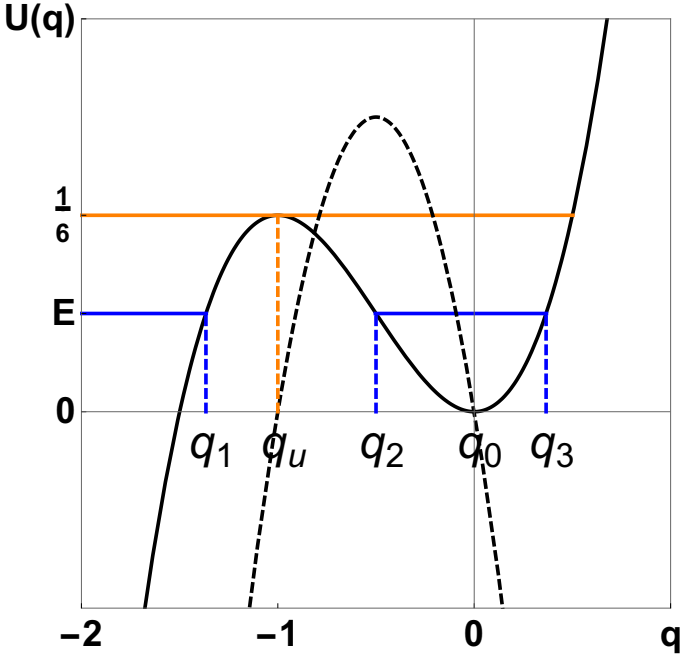
A.3 Plot of Jacobi elliptic functions

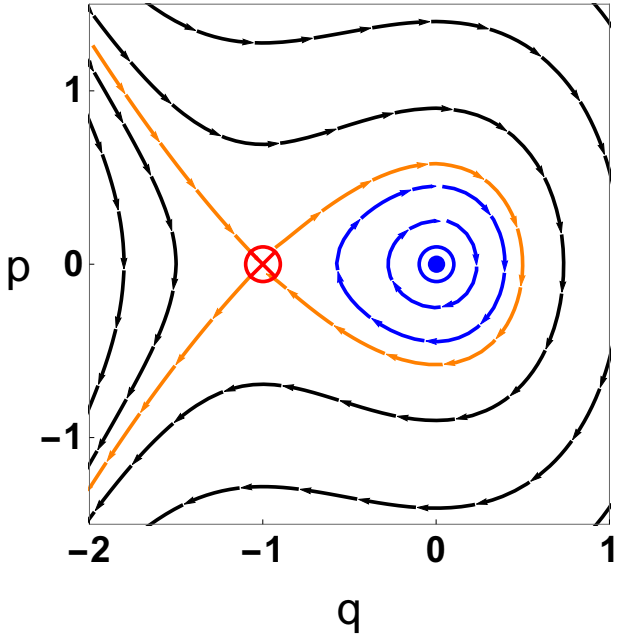
$$\mathrm{sn}^2 t + \mathrm{cn}^2 t = 1 \qquad\qquad k^2 \mathrm{sn}^2 t + \mathrm{dn}^2 t = 1$$

	k=0	k=0.5	k=0.95	k=1
sn(t,k)				
cn(t,k)				
dn(t,k)				

1D Sextupole, $H[p, q; t] = \frac{p^2}{2} + \frac{q^2}{2} + \frac{q^3}{3}$

There are two stationary points: elliptic ($q_{\mathrm{s}} = 0$) and hyperbolic ($q_{\mathrm{u}} = -1$), corresponding to a local minimum $U(0) = 0$ and local maximum $U(-1) = \frac{1}{6}$ of the potential.





Distinct topologies in a phase space

Oscillations Closed curves corresponding to nonlinear oscillations with $0 < E < \frac{1}{6}$ around stable equilibrium point q_{s} and isolated by homoclinc orbit with $E = \frac{1}{6}$ attached to point q_{u}.

Scattering 1 For trajectories with $E > \frac{1}{6}$, a particle's energy is large enough to escape from minimum of potential and they all scatter to $-\infty$ from the right side of the barrier.

Scattering 2 In addition to homoclinc orbit and unstable stationary point, level set $E = \frac{1}{6}$ contains two separatrices

$$p_{\mathrm{sep}}(q) = \pm(1+q)\sqrt{\frac{1}{3}\left(\frac{1}{2} - q\right)}, \qquad q < -1.$$

They separate scattered particles with large energy from particles with lower energy $E < \frac{1}{6}$ which cannot go through the left side of the potential.

Finite trajectories, $0 < E < \frac{1}{6}$

$$\mathrm{d}t = \pm\sqrt{\frac{3}{2}}\frac{\mathrm{d}q}{\sqrt{(q_3 - q)(q - q_2)(q - q_1)}}, \qquad q_1 < q_2 \leq q \leq q_3$$

where q_i $(i = 1, 2, 3)$ in the rhs of the equation are obtained by solving for stop points $U(q) = E$. The use of substitution

$$q = q_2 + (q_3 - q_2)\cos^2\phi(t) = q_3 - (q_3 - q_2)\sin^2\phi(t)$$

reduces right side of equation to familiar form of elliptic integral

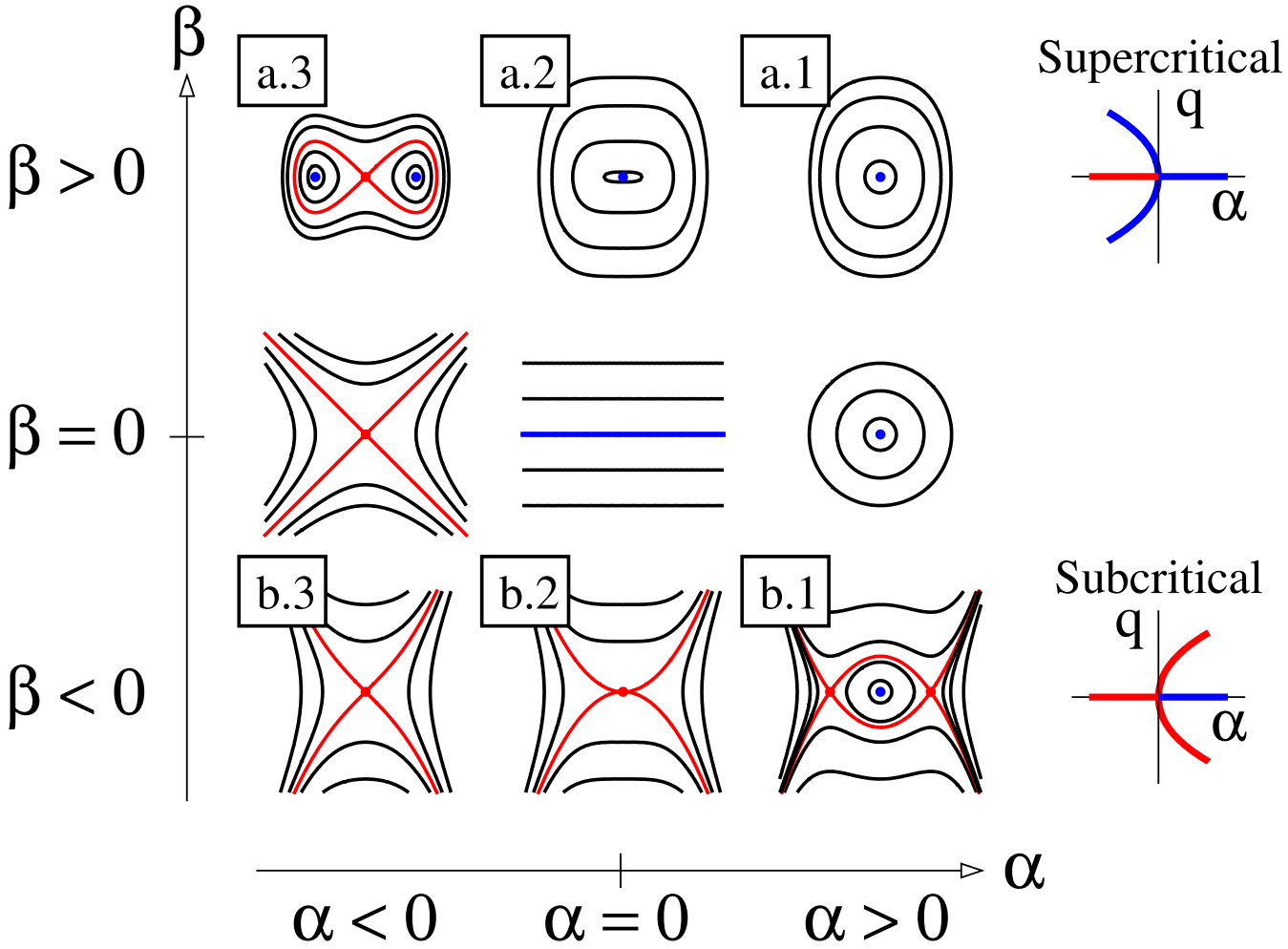
$$\mathrm{d}t = \mp\sqrt{\frac{6}{q_3 - q_1}}\frac{\mathrm{d}\phi}{\sqrt{1 - k^2\sin^2\phi}}, \qquad k = \sqrt{\frac{q_3 - q_2}{q_3 - q_1}}.$$

$$
\begin{aligned}
T &= 2\sqrt{\frac{6}{q_3 - q_1}}\, K(k), \\
\phi(t) &= \mathrm{am}\left(\sqrt{\frac{q_3 - q_1}{6}}\, t, k\right).
\end{aligned}
$$

The last equation gives dynamical variables as a function of time

$$
\begin{aligned}
q(t) &= q_2 + (q_3 - q_2)\,\mathrm{cn}^2\left(\sqrt{\frac{q_3 - q_1}{6}}\, t, k\right) \\
p(t) &= -2(q_3 - q_2)\sqrt{\frac{q_3 - q_1}{6}} \times \\
&\quad \times \mathrm{sn}\left(\sqrt{\frac{q_3 - q_1}{6}}\, t, k\right) \mathrm{cn}\left(\sqrt{\frac{q_3 - q_1}{6}}\, t, k\right) \mathrm{dn}\left(\sqrt{\frac{q_3 - q_1}{6}}\, t, k\right)
\end{aligned}
$$

1D Octupole, $H[p, q; t] = \frac{p^2}{2} + \alpha \frac{q^2}{2} + \beta \frac{q^4}{4}$



Case a.1

The potential has only one minimum $q_0 = 0$ since $\partial\, U(q)/\partial q = 0$ has only one real root. This minima corresponds to elliptic stationary point with $U(q_0) = 0$. All motions are stable and particles perform nonlinear oscillations around q_0 with energy spectra $E \geq 0$. Stop points are given by equation

$$E - U(q) = -\frac{1}{4}(q^4 + 2\,q^2 - 4\,E) = 0.$$

It has two real roots we are looking for

$$q_{1,2} \equiv q_{\pm} = \pm\left(\sqrt{4\,E + 1} - 1\right)^{1/2}$$

and two imaginary roots

$$q_{3,4} = \pm\, i\left(\sqrt{4\,E + 1} + 1\right)^{1/2}.$$

Using symmetries between roots

$$q_1 = -q_2, \qquad q_3 = -q_4, \qquad q_{1,2}^2 + q_{3,4}^2 = -2,$$

where in the last relation any pair of indecies is appropriate, one can rewrite the polynomial as

$$\begin{aligned} -\frac{1}{4}(q^4 + 2\,q^2 - 4\,E) &= -\frac{1}{4}(q - q_1)(q - q_2)(q - q_3)(q - q_4) \\ &= \frac{1}{4}(q_+^2 - q^2)(q^2 + q_+^2 + 2). \end{aligned}$$

The use of substitution $q(t) = q_+ \cos\varphi(t)$ gives us

$$\mathrm{d}t = \mp \frac{1}{\sqrt{1 + q_+^2}} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \qquad k = \frac{q_+}{\sqrt{2\,(1 + q_+^2)}}.$$

Integration gives

$$
\begin{aligned}
T &= \frac{4}{\sqrt{1+q_+^2}} K(k), \\
\phi(t) &= \mathrm{am}\left(\sqrt{1+q_+^2}\, t, k\right).
\end{aligned}
$$

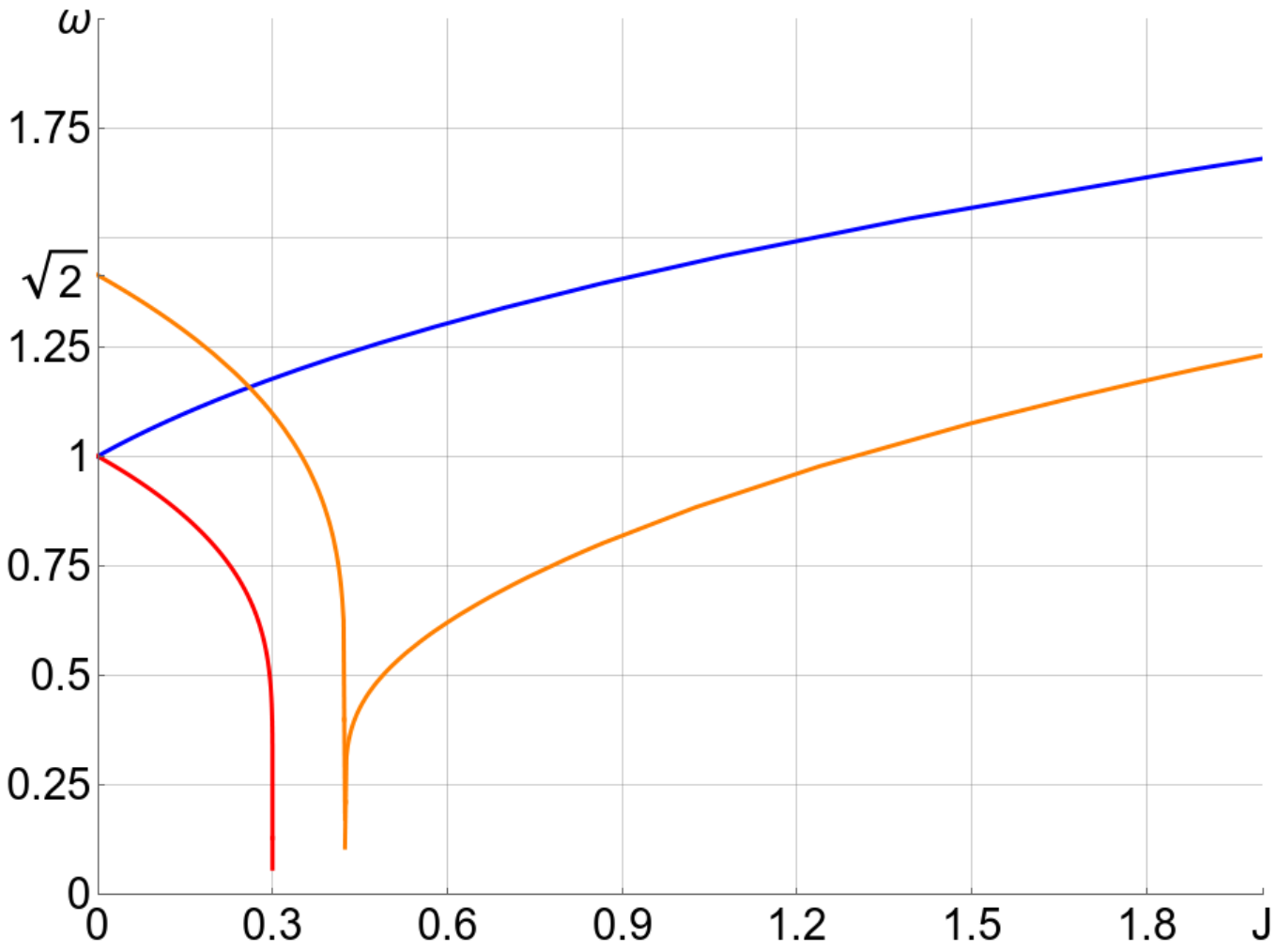
Since

$$\cos \mathrm{am}(u,k) = \mathrm{cn}(u,k)$$

oscillations are simply Jacobi cosine functions and equations of motion are

$$
\begin{aligned}
q(t) &= q_+ \,\mathrm{cn}(\sqrt{1+q_+^2}\, t, k), \\
p(t) &= -q_+^2 \,\mathrm{sn}(\sqrt{1+q_+^2}\, t, k)\,\mathrm{dn}(\sqrt{1+q_+^2}\, t, k).
\end{aligned}
$$

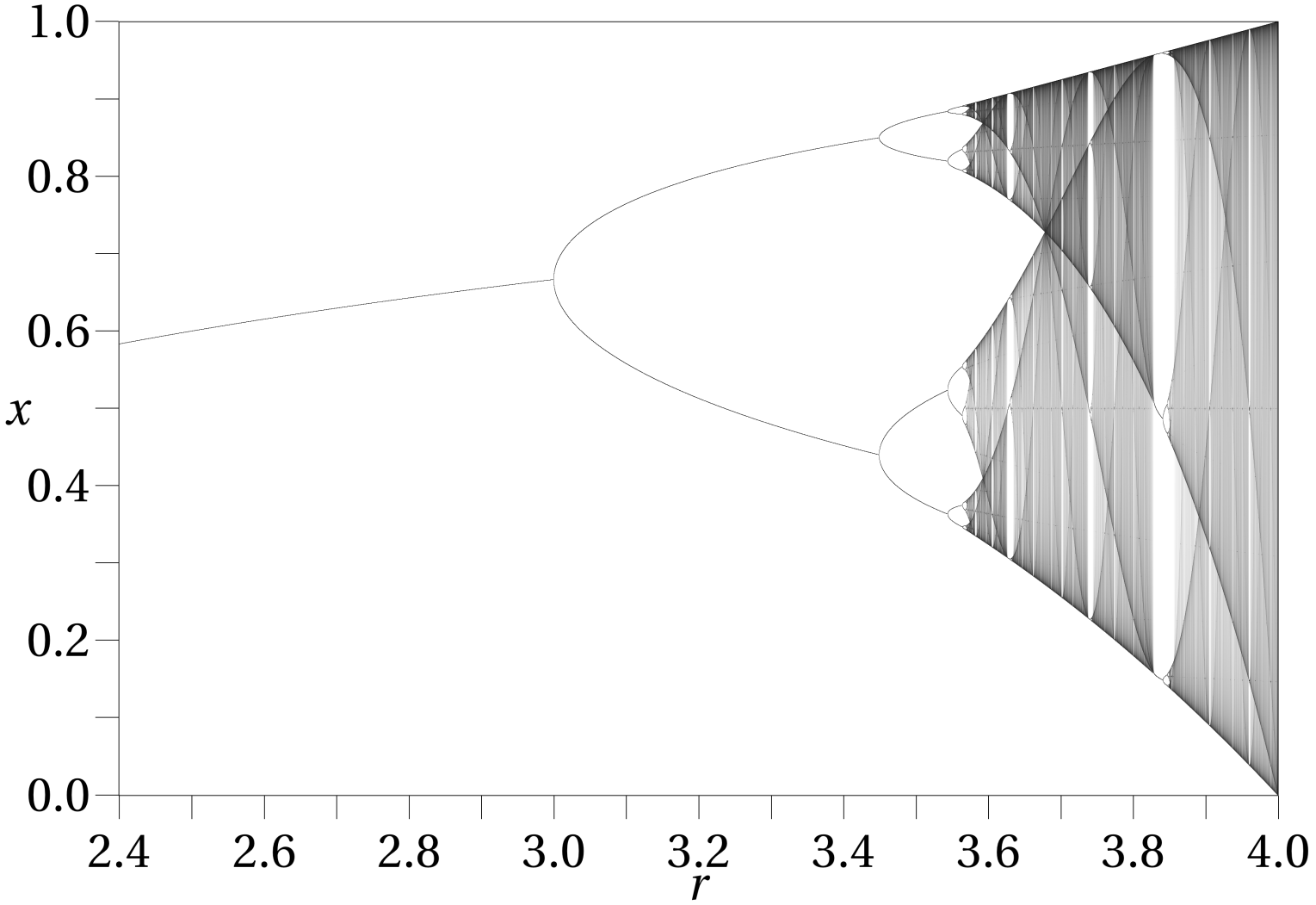
Action angle variables



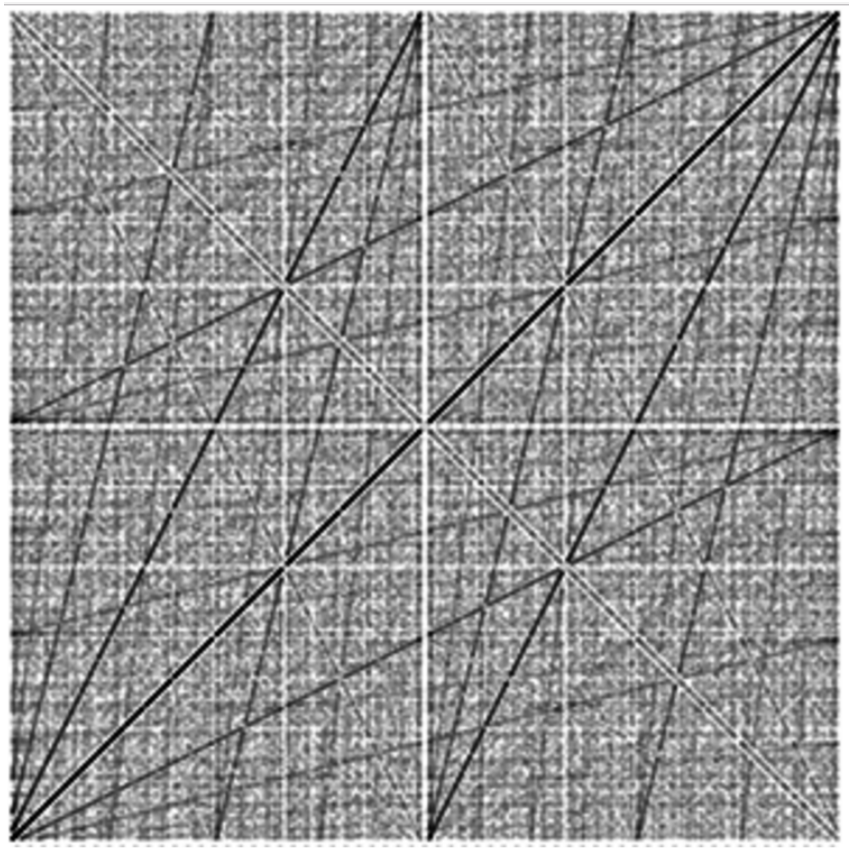
Part II

Discrete Dynamics

1. Logistuc map, $x_{n+1} = r\,x_n\,(1 - x_n)$

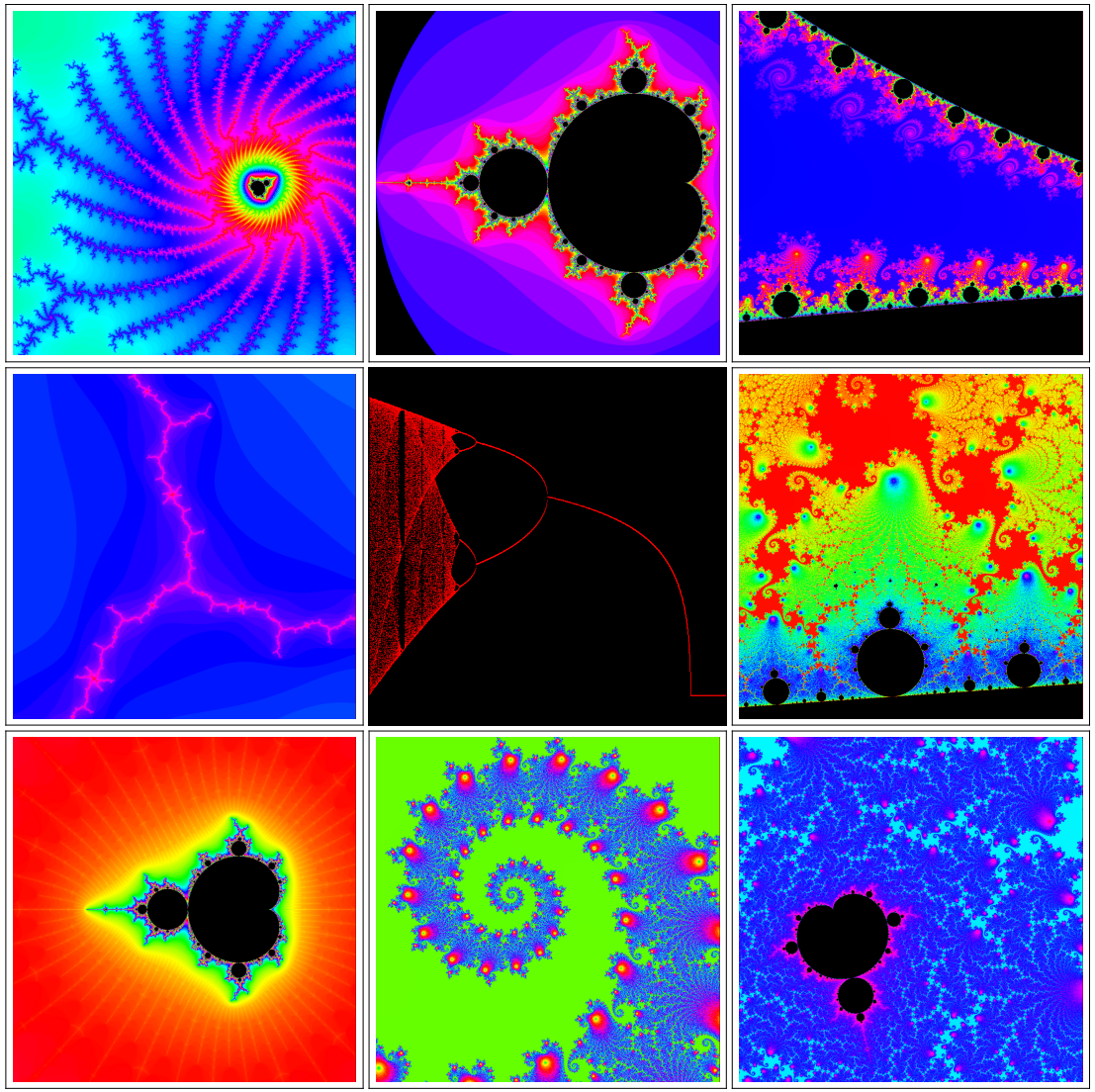


2. Dyadic map/bit shift map/Bernouli map/doubling map or sawtooth map, $x_{n+1} = 2\,x_n \bmod 1$

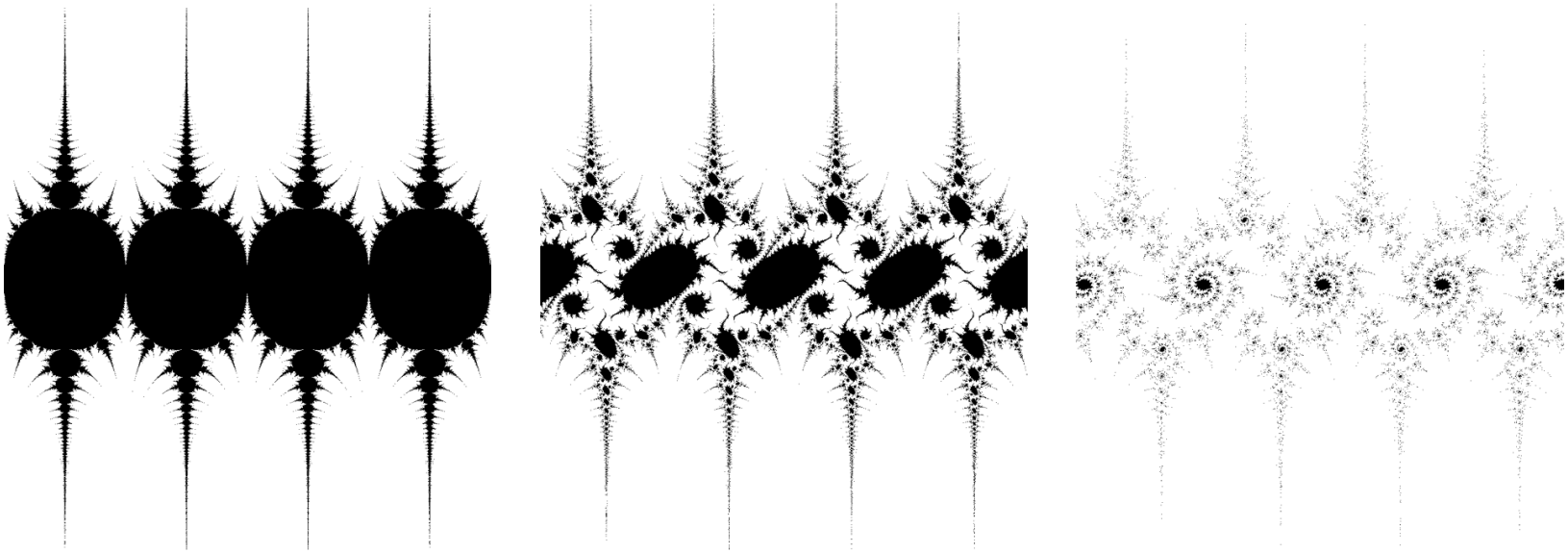
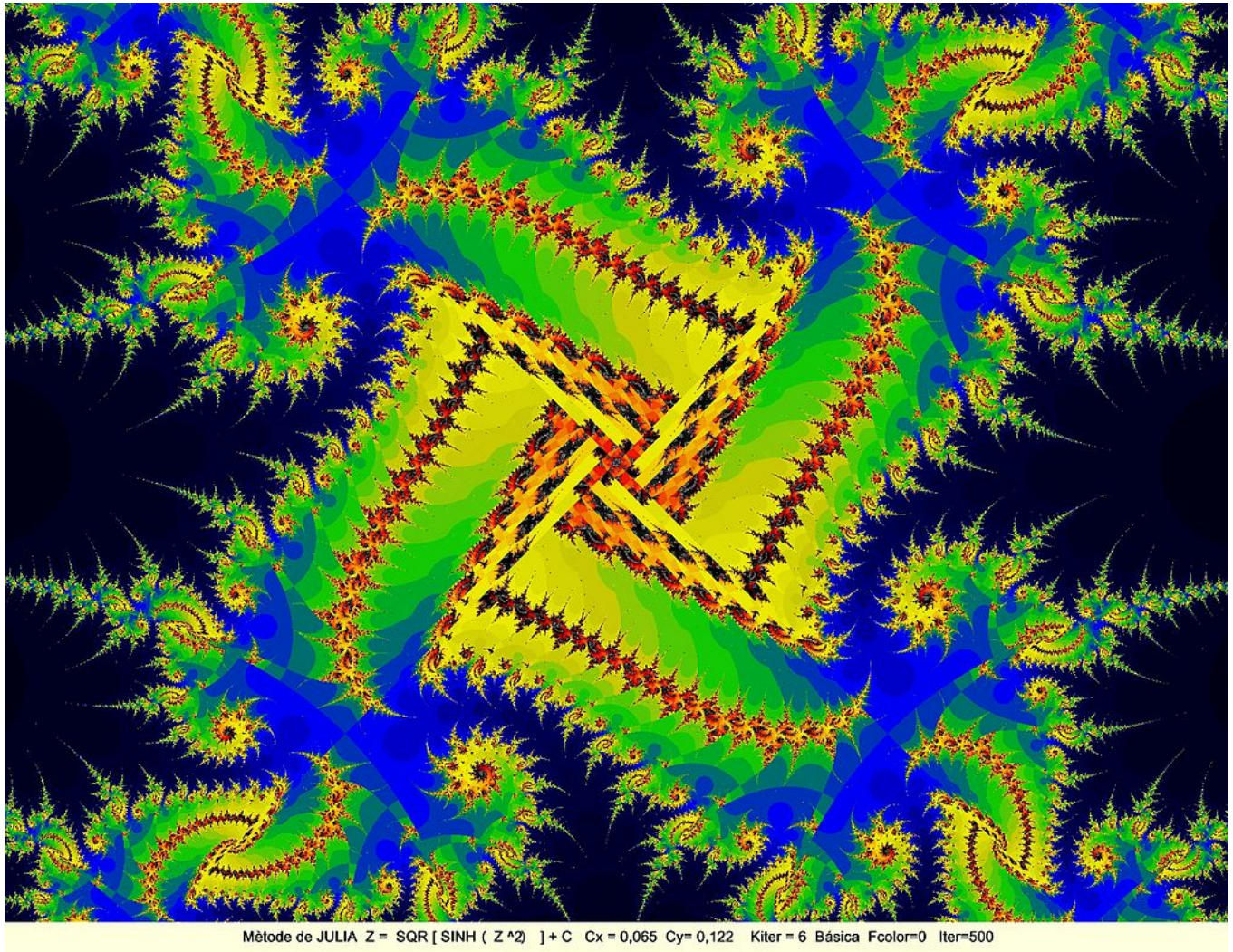


Special case of Logistic map for $r = 4$ with $x \to \sin^2(2\,\pi\,x)$.

3. Mandelbrot Set, $z_{n+1} = z_n^2 + c$



4. Julia and Fatou sets, $z_{n+1} = c\, sin(z)$



5. Julia and Fatou sets