Integrable Particle Dynamics in Accelerators

Lecture 1: Introduction

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Fermilab

Welcome

- Upon completion of this course, you are expected to understand the basic principles that underline the physics of integrable particle dynamics in particle accelerators. You will learn the notion of integrability as referring to the existence of the adequate number of invariants or constants of motion. Applying this knowledge, you will develop an insight into the mechanisms of chaotic and non-chaotic particle motion in accelerators, action-phase variables, examples of nonlinear focusing systems leading to integrable motion, and nonlinear beam-beam effects.
- Credit Requirements
  - Evaluated based on the following performances: Final exam (50%), Homework assignments and class participation (50%).

http://home.fnal.gov/~nsergei/USPAS_W2019/

We will post our lecture notes after each half-day session

Monday Jan 28, 2019

9:00 to 10:15 Part 1: Introduction

10:15 to 10:30 Break

10:30 to 12:30 Part 2: Particle Accelerators

1. Courant-Snyder invariant
3. Betatron tune

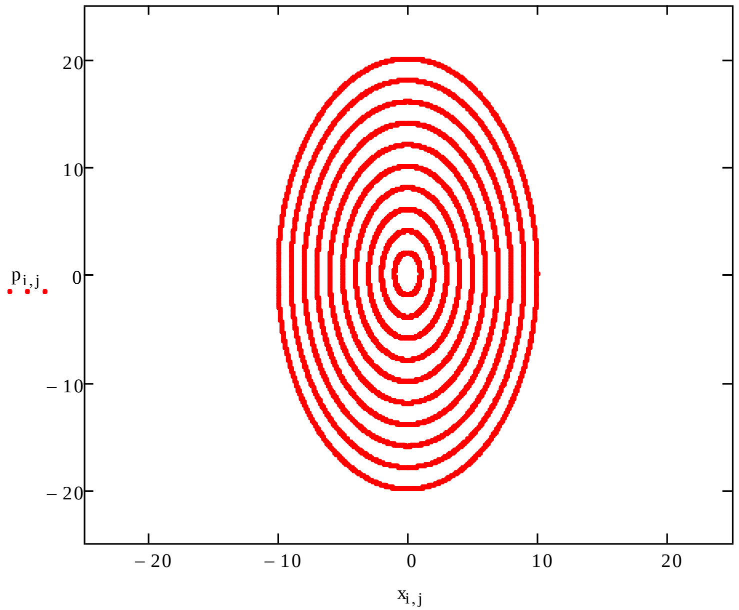
12:00 to 13:30 Lunch

Harmonic oscillator - an example of a conservative system

$$m\ddot{x} = -kx$$

$$x(t) = x_o \sin\left(\sqrt{\frac{k}{m}}t + \phi_0\right)$$

$$p(t) = m\dot{x} = x_o\sqrt{mk}\cos\left(\sqrt{\frac{k}{m}}t + \phi_0\right)$$

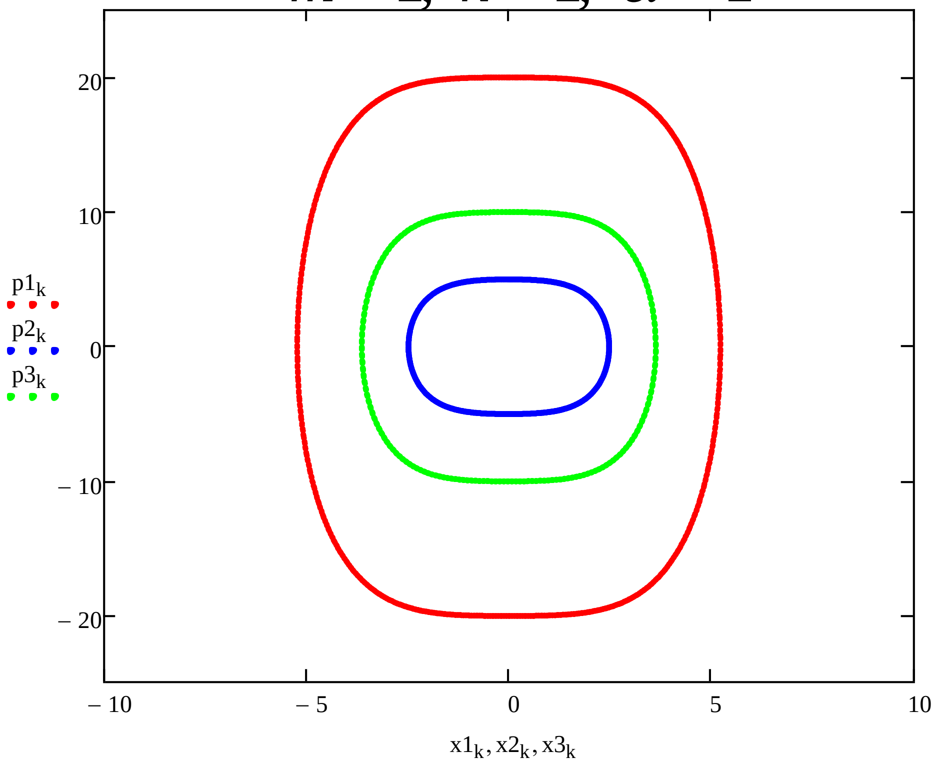


- The total energy is conserved
- Motion is finite and stable for all initial conditions
  - Small variations in initial conditions keeps trajectories near each other
- All trajectories are bounded and have the same period (isochronous)

A non-linear oscillator

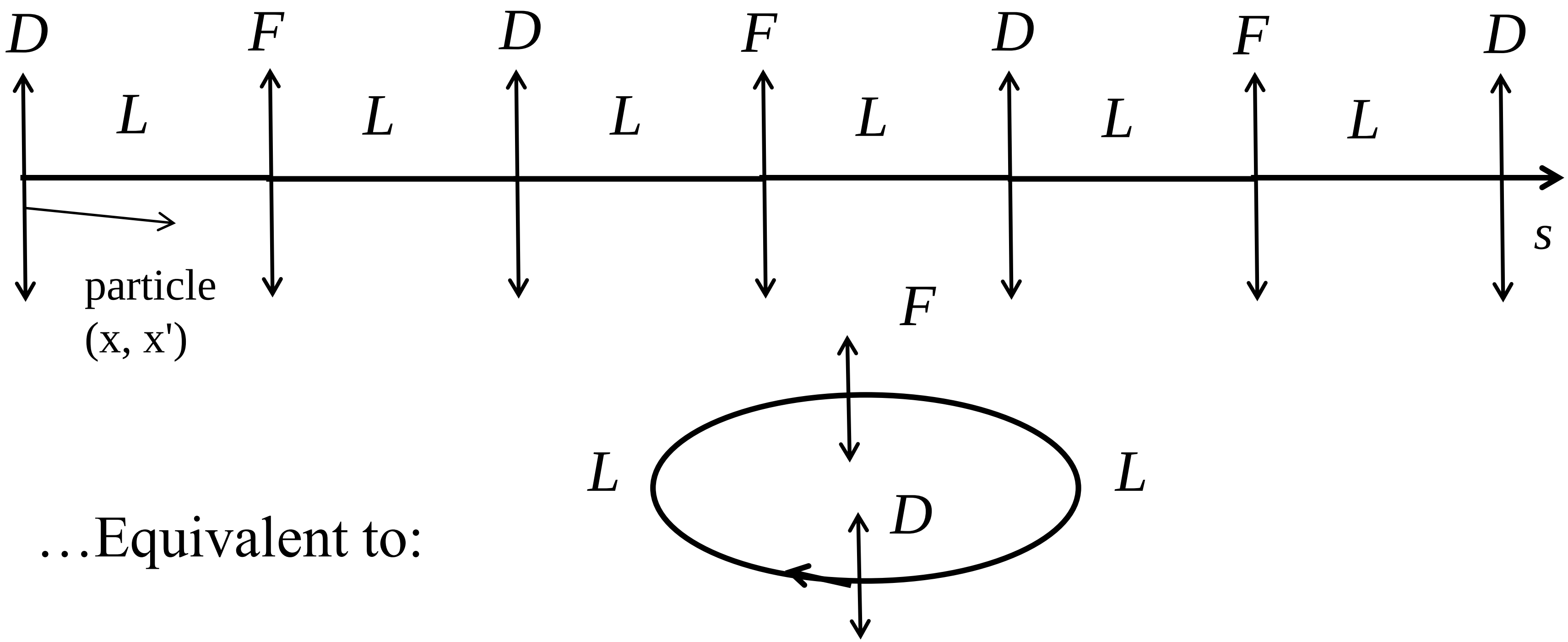
$$m\ddot{x} = -kx - \alpha x^3$$

$$m = 1;\ k = 1;\ \alpha = 1$$



- Equations of motion can be solved in Jacobi elliptic functions
- The total energy is conserved
- Motion is finite and stable for all initial conditions
- All trajectories have different periods, depending on initial conditions
  - Motion is bounded but non-isochronous

A simple periodic focusing channel (FODO)



…Equivalent to:

- Thin alternating lenses and drift spaces
- Consider a one-turn map, $\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \mathbf{M} \begin{pmatrix} x \\ x' \end{pmatrix}_0$
- Track a particle for "many" turns, n

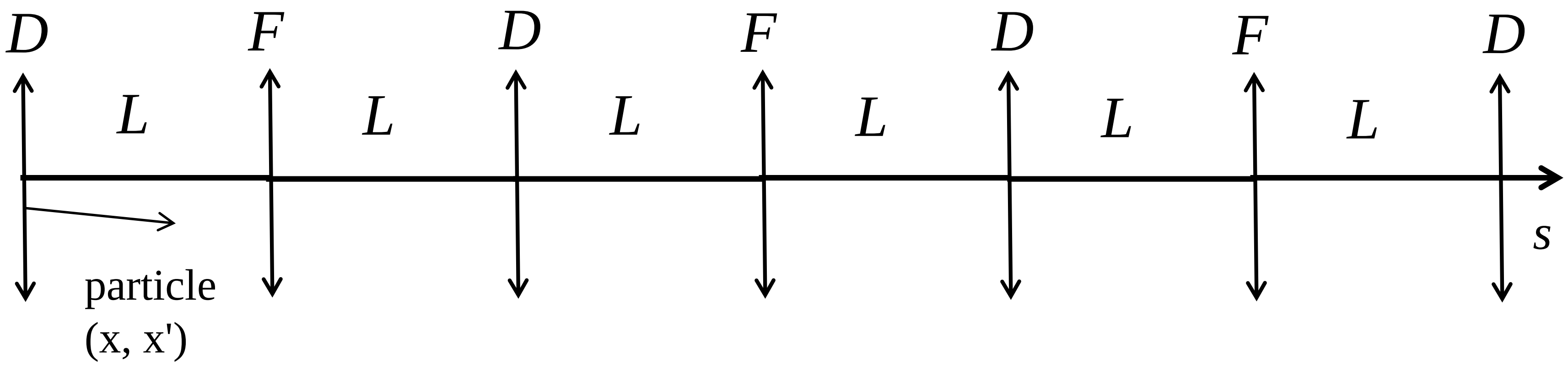
Simplest accelerator elements

- A drift space: L – length $\begin{pmatrix} x \\ x' \end{pmatrix}_L = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$

- A thin focusing lens: $\begin{pmatrix} x \\ x' \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$

- A thin defocusing lens: $\begin{pmatrix} x \\ x' \end{pmatrix}_D = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$

Particle motion in accelerator

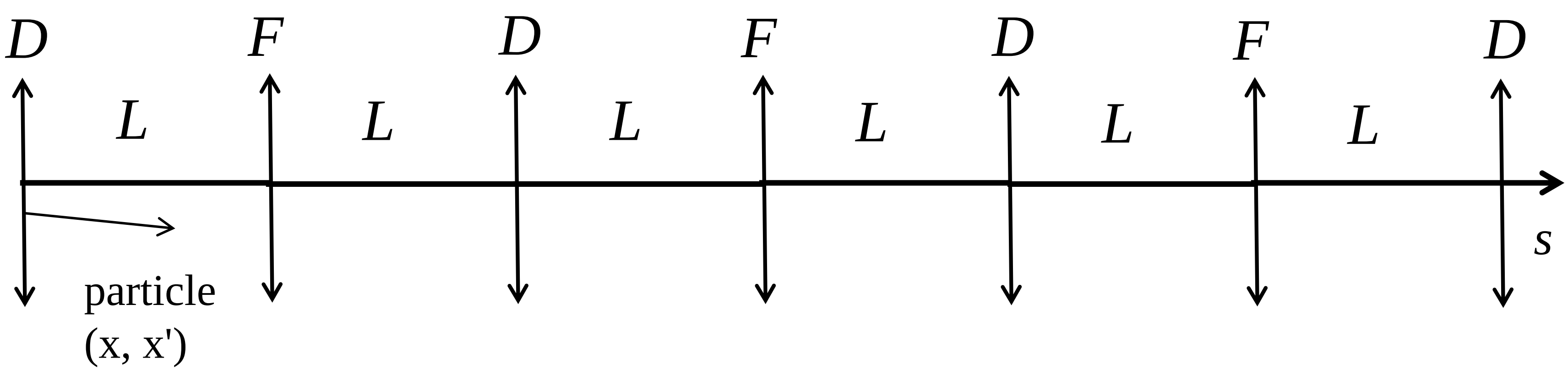


- Let's launch a particle with initial conditions x and x'

- Questions:
  1. Is the particle motion periodic in (x, x')?
  2. Is the particle motion stable (bounded)?

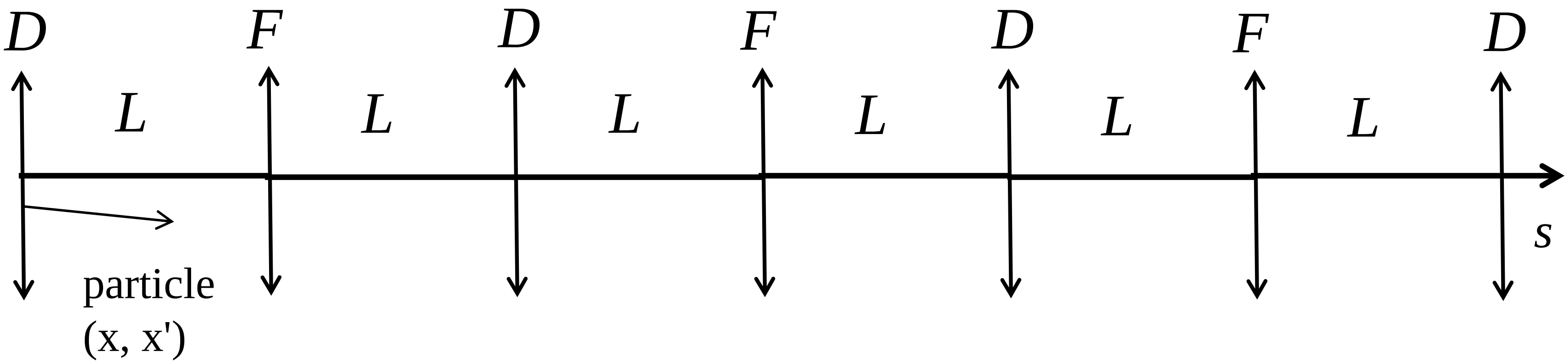
Particle stability in a simple channel

- Q1: Possible answers
  - Yes or no
- Q2: Possible answers:
  A. Always stable
  B. Stable only for some initial conditions
  C. Stable only for certain *L* and *F*



Particle stability in a simple channel

- Correct answers: (Q1: yes, for with C), Q2:
  - ~~A. Always stable~~
  - ~~B. Stable only for some initial conditions~~
  - C. Stable only for certain *L* and *F* → $\boxed{0 < \frac{L}{F} < 2}$



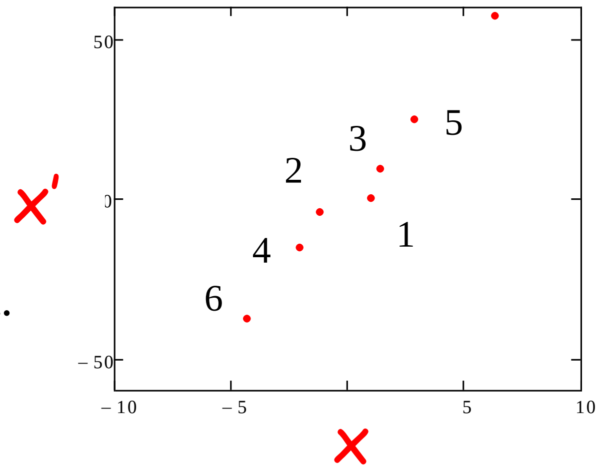
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \underbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ F^{-1} & 1 \end{pmatrix}}\begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

$$\text{Stability: } \left|\mathrm{Trace}(\mathbf{M})\right| < 2$$

Phase space trajectories: x' vs. x

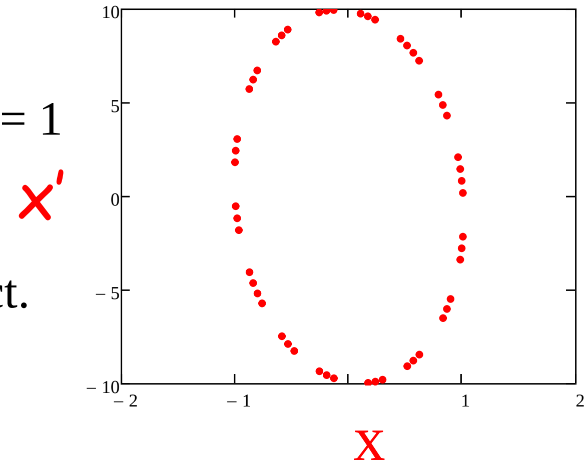
$F = 0.49$, $L = 1$

7 periods,
unstable traject.



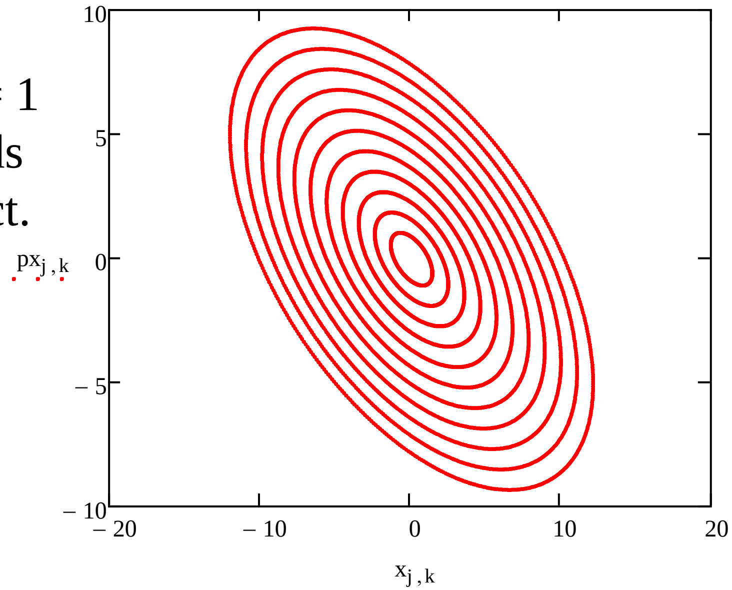
$F = 0.51$, $L = 1$

50 periods,
stable traject.



When this simple focusing channel is stable, it is stable for ALL initial conditions !

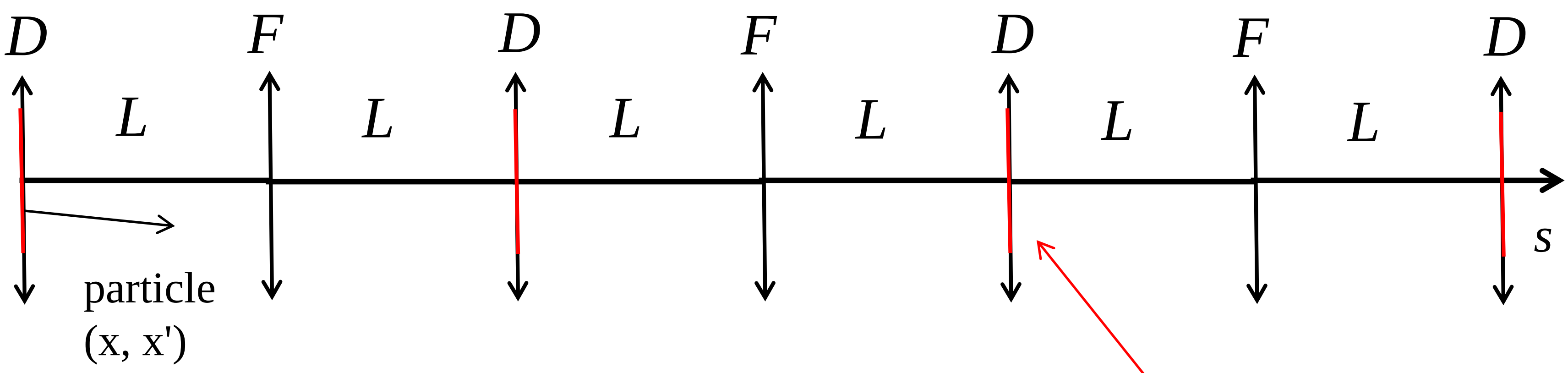
$F = 1.2$, $L = 1$
1000 periods
stable traject.



All trajectories are periodic and, ALL particles are isochronous: they oscillate with the same frequency (betatron tune)!

$$2\pi\nu = \operatorname{acos}\left(\frac{\mathrm{Tr}(\mathbf{M})}{2}\right)$$

Let's add a cubic nonlinearity…

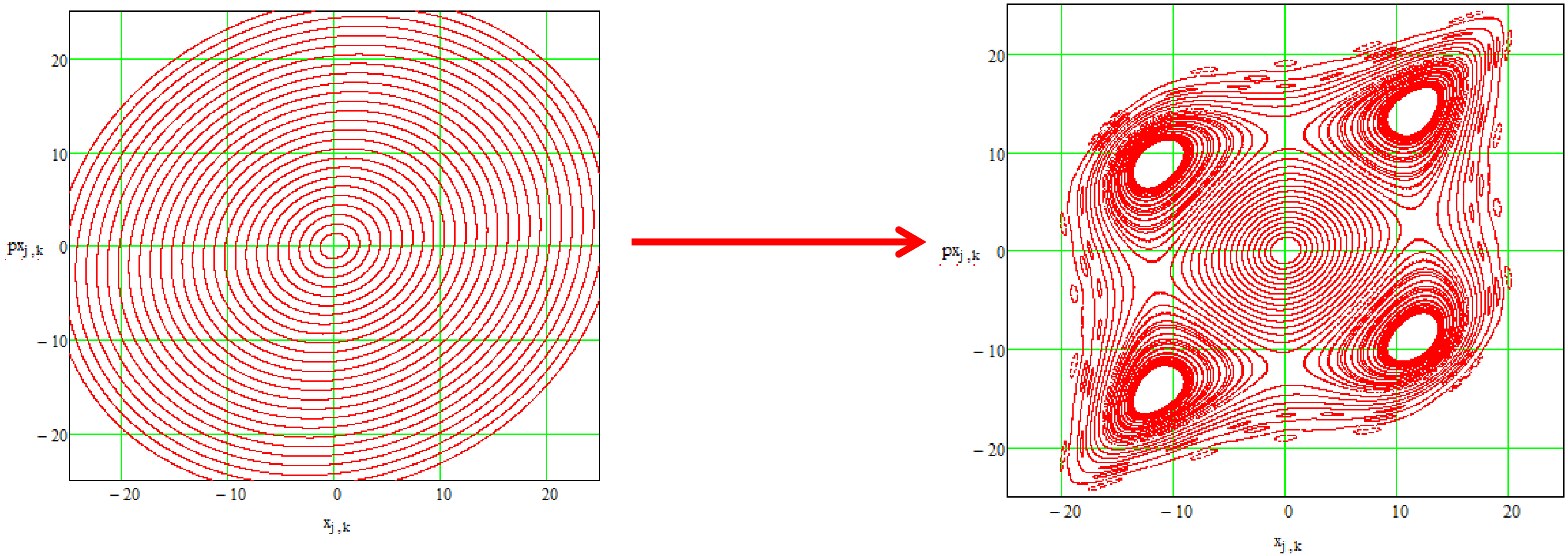


add a cubic
nonlinearity
in every D lens

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} x \\ x' - \alpha x^3 \end{pmatrix}_i$$

The result of this nonlinearity:

- Betatron oscillations are no longer isochronous:
  - The frequency depends on particle amplitude (initial conditions)
- Stability depends on initial conditions
  - Regular trajectories for small amplitudes
  - Resonant islands (for larger amplitudes)
  - Chaos and loss of stability (for even larger amplitudes)



Discussion

- Let's discuss the 4 cases we considered:

1. A simple harmonic oscillator
2. An oscillator with a cubic nonlinearity
3. A particle in a FODO channel
4. A particle in a FODO channel with a cubic nonlinearity

- What are their similarities and differences?

Discussion continued

- Cases 1 and 2 are both conservative systems, i.e. the total energy is conserved
- Cases 1 and 3 are both isochronous systems, i.e. all particles have the same frequencies.
- Case 3 is NOT a conservative system, but it has a conserved quantity (i.e. the integral of motion), called the Courant-Snyder invariant
  - This is an example of a linear INTEGRABLE system

$$I_x = cx^2 + (d-a)xx' - bx'^2 \qquad \text{for} \qquad \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Case 4 is similar to Case 3 for some initial conditions.
  - This is an example of a nonlinear NON-integrable system

Frequencies

1. A harmonic oscillator: $$\omega_0 = \sqrt{\frac{k}{m}}$$

2. A non-linear (cubic) oscillator: $$\omega(A) \approx \omega_0 + \frac{3\alpha}{8\omega_0} A^2$$

3. A linear integrable accelerator map: $$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \mathbf{M} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

The betatron tune (the rotation number): $$\nu = \frac{1}{2\pi} \operatorname{acos}\left(\frac{\operatorname{Trace}(\mathbf{M})}{2} \right)$$

4. A non-linear non-integrable accelerator map:

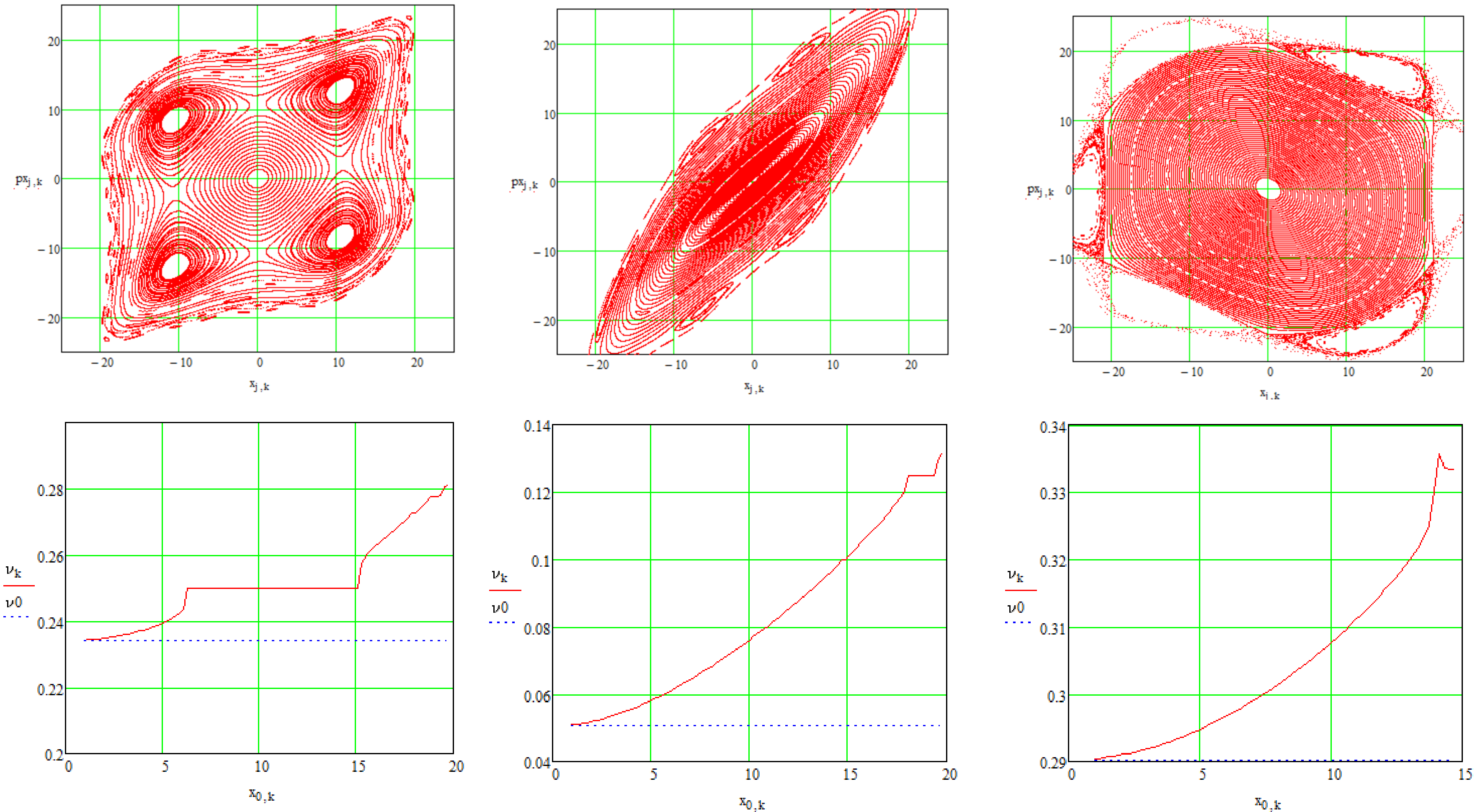
$$\nu = \lim_{N \to \infty} \frac{1}{2\pi N} \sum_{n=1}^{N} R_n$$

R_i is the rotation angle between points *i* and *i-1* around the fixed point

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} x \\ x' - \alpha x^3 \end{pmatrix}_i$$

Examples

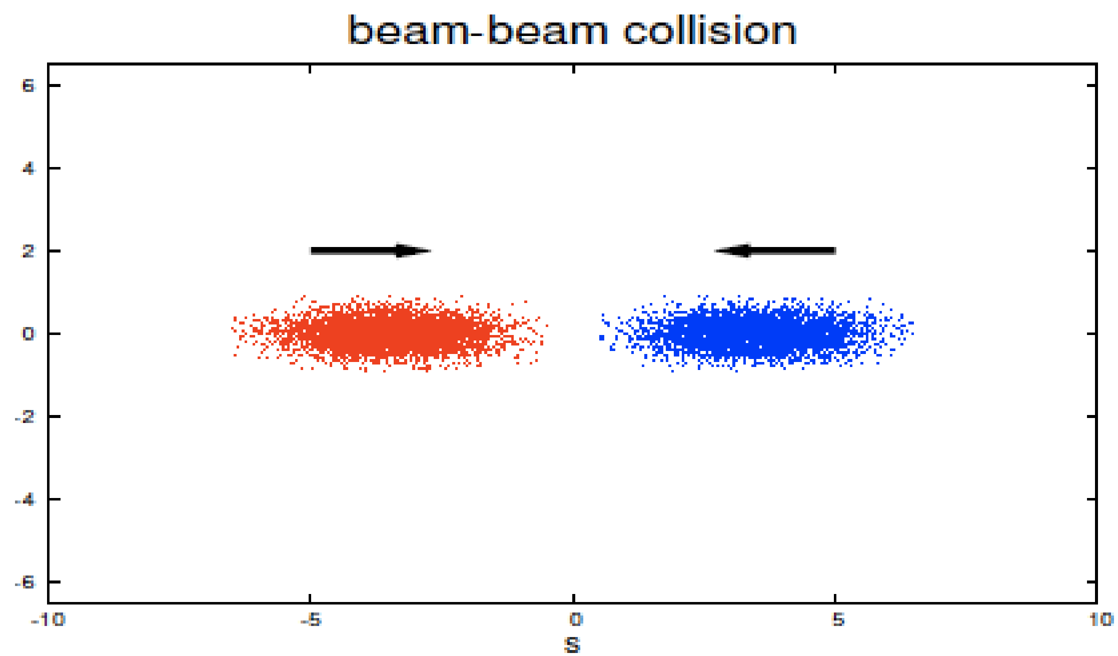
$a = 0.2$ | $a = 1.9$ | $a = -0.5$



Common features of non-integrable systems: mode locking

Example: beam-beam effects

- Beams are made of relativistic charged particles and represent an electromagnetic potential for other charges



- Typically:
  - 0.001% (or less) of particles collide
  - 99.999% (or more) of particles are distorted

Beam-beam effects

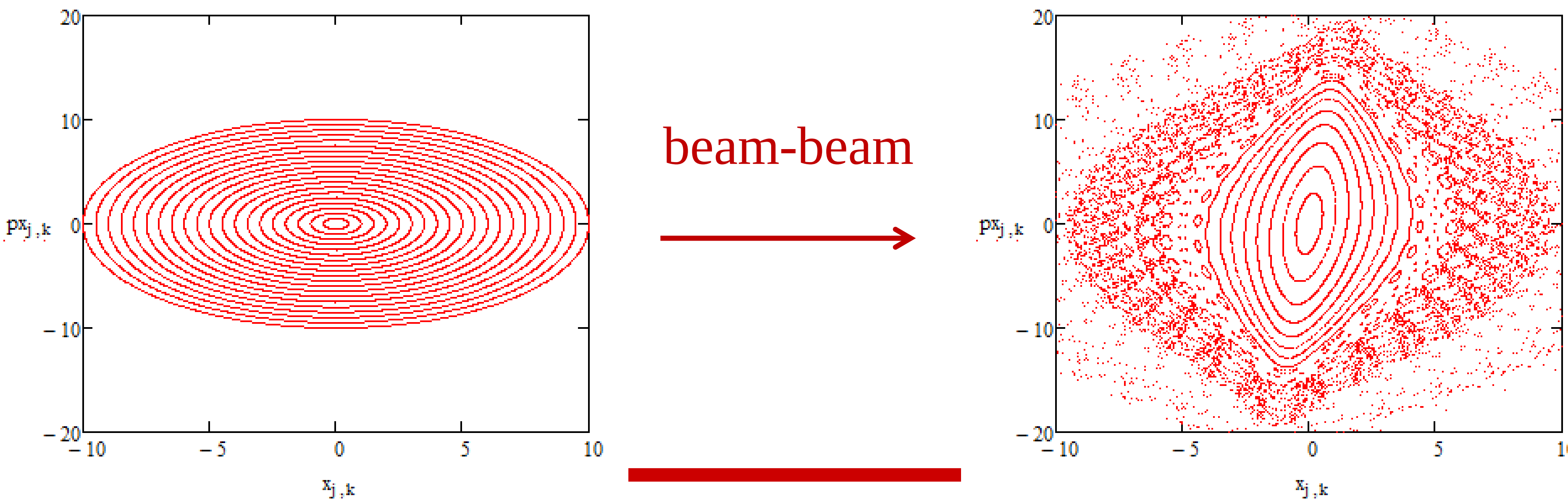
- One of most important limitations of all past, present and future colliders

Luminosity

$$\mathcal{L} \propto \frac{N_p^2}{\sigma_x \sigma_y} \cdot n_b$$

Beam-beam Force

$$F \propto \frac{N_p}{\sigma} \cdot \frac{1}{r} \cdot \left[1 - e^{-\frac{r^2}{2\sigma^2}}\right]$$



Example: electron storage ring light sources

- Low beam emittance (size) is vital to light sources
  - Requires Strong Focusing
  - Strong Focusing leads to strong chromatic aberrations
  - To correct Chromatic Aberrations special nonlinear magnets (sextupoles) are added

dynamic aperture limitations lead to reduced beam lifetime

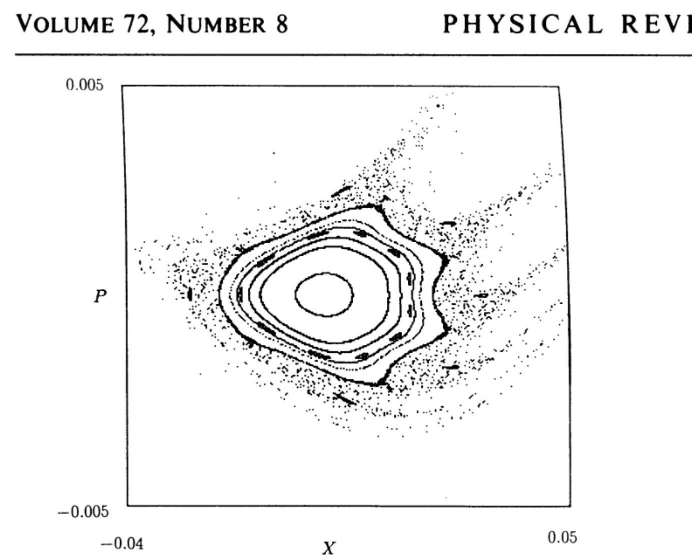


FIG. 1. Surface of section for the ALS.

Example: Landau damping

COLLIDING BEAMS: PRESENT STATUS; AND THE SLAC PROJECT*

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The discovery in the early '60's at the Princeton-Stanford ring of what was thought to be the resistive wall instability brought the realization that circular accelerators are fundamentally unstable devices because of the interaction of the beam with its environment. Stability is achieved only through Landau damping and/or some external damping system.

Report at HEAC 1971

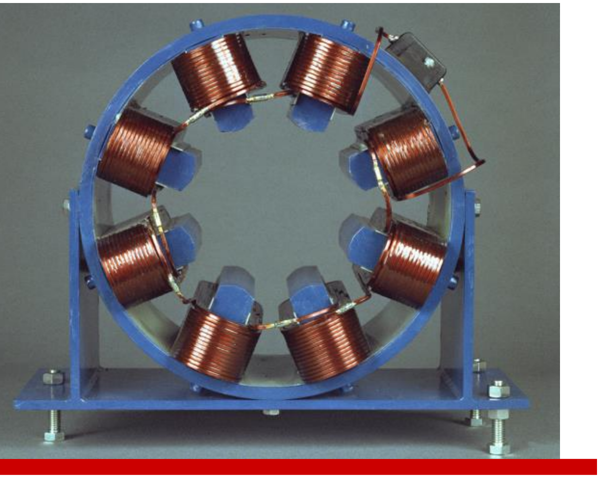
- Landau damping – the beam's "immune system". It is related to the spread of betatron oscillation frequencies. The larger the spread, the more stable the beam is against collective instabilities.
  - The spread is achieved by adding special magnets -- octupoles
- External damping (feed-back) system – presently the most commonly used mechanism to keep the beam stable.

Most accelerators rely on both

LHC:

- Has a transverse feedback system
- Has 336 Landau Damping Octupoles

- Octupoles (an 8-pole magnet):
  - Potential: $\varphi(x,y) \propto x^4 + y^4 - 6x^2y^2$
  - Results in a cubic nonlinearity (in force)



Summary I

- Chaotic and unstable particle motion appears even in simplest examples of accelerator focusing systems with nonlinearities
  - The nonlinearity shifts the particle betatron frequency to a resonance ($n\omega = k$)
  - The same nonlinearity introduces a time-dependent resonant kick to a resonant particle, making it unstable.
    - The nonlinearity is both the driving term and the source of resonances simultaneously

Summary II

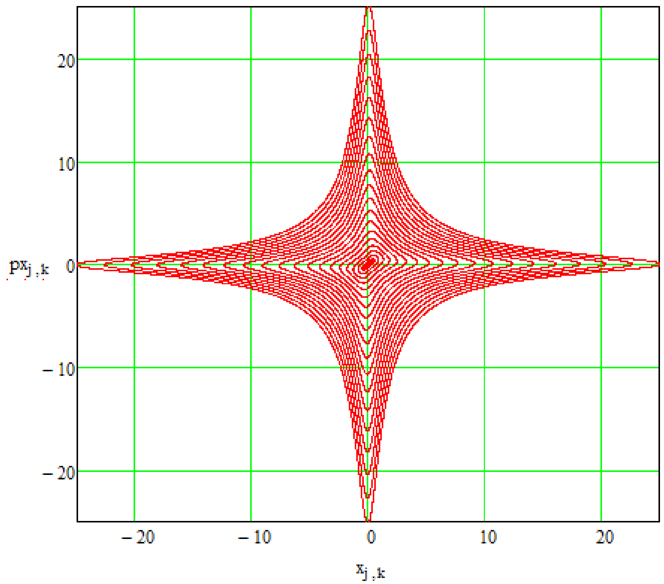
- The existence of conserved quantities or integrals of motion (the total energy in Cases 1&2, the C-S invariant in Case 3) in general, is the necessary condition for a particle not to exhibit a chaotic behavior. "Good" integrals of motion confine particle motion to a finite region.
  - It may not be sufficient, because there may not exist a sufficient number of these integrals of motion to fully confine the particle motion.
  - Or, the integrals themselves may allow particles to wander to infinity (which is bad specifically for accelerators).

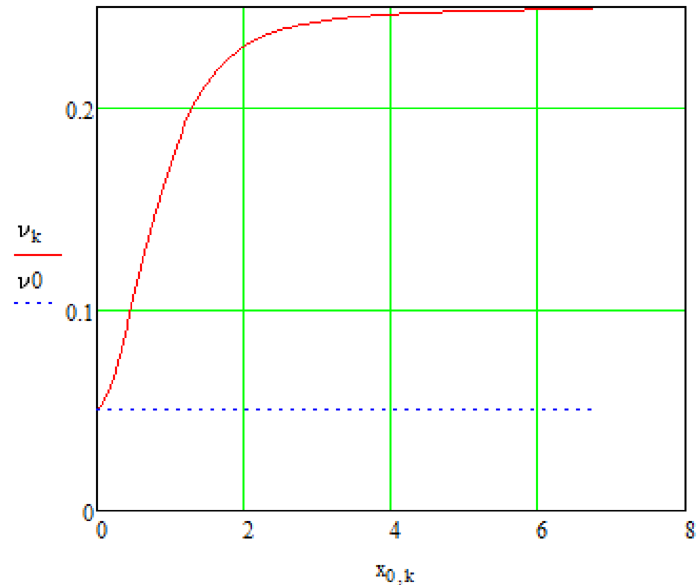
Integrability in Accelerators

- All present machines are designed to be integrable: drifts, quadrupoles, dipoles-- can all be accommodated in the Courant-Snyder invariants.
  - These are all examples of linear systems (equivalent to a harmonic oscillator)
- The addition of nonlinear focusing elements to accelerators breaks the integrability, …but this additions are necessary in all modern machines – for chromatic corrections, Landau damping, strong beam-beam effects, space-charge, etc
  - Non-integrable – “bad”, integrable – “good” (but not always…)
- Question: are there such “magic” nonlinearities that do not break the integrability?
  - Yes, but the number of examples is still very limited.

Example: McMillan integrable mapping

$a = 1.9;\ b = 1$





$$x_i = p_{i-1}$$

$$p_i = -x_{i-1} + \frac{a p_{i-1}}{b p_{i-1}^2 + 1}$$

Integral of motion:

$$I_x = b x^2 p^2 + x^2 + p^2 - a x p$$

- Note:
  - No resonant islands, no mode locking
  - No chaos

Accelerator research areas, where integrability would help

- Single particle dynamics:
  1. How to make the dynamical aperture larger? (light sources, colliders)
  2. How to make the tune spread larger? (Landau damping in high-intensity rings)
  3. How to reduce beam halo?
- Multi-particle dynamics:
  1. How reduce detrimental beam-beam effects?
  2. How to compensate space-charge effects?
  3. How to suppress instabilities?
  4. How to reduce beam halo?